Technical Appendix to ``Effects of a Money-financed Fiscal Stimulus Without Irredeemability of Money''

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1 The Model

Folowing Benigno and Benigno (2003, ECB-WP), we extend the model in Gali (2020) to a two-country economy model.

1.1 Households

We consider a world economy populated by a measure one of households. The population on the segment $[0,\nu)$ belongs to the Home country (*H*) while the one on the segment $[\nu,1]$ belongs to the Foreign country (*F*).

Households' preference is given by:

$$\sum_{t=0}^{\infty}\beta^t U_t^j \text{ , (A-1-1)}$$

with

$U_t^j \equiv \left\{U\left(C_t^j, L_t^j\right) - V\left(N_t^j\right)\right\} Z_t$ if $j \in [0,\nu)$ and $U_t^{*,j} \equiv \left\{U\left(C_t^{*,j}, L_t^{*,j}\right) - V\left(N_t^{*,j}\right)\right\} Z_t$ if

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$j \in [\nu, 1]$,

where the index *j* denotes a variable that is specific to household *j* and the index *i* denotes a variable specific to the country *H* or *F* in which *j* resides. To clarify the notation that follows *i* will be replaced by a star when referring to country *F* and will be suppressed when referring to country *H*. (Nominal) Households' budget constraints in units of countries *H* and *F* currencies are given by:

$$\int_0^{\nu} P_t(h) C_t^j(h) dh + \int_{\nu}^1 P_t(f) C_t^j(f) df + D_{H,t}^j + D_{F,t}^j E_t + M_t^j$$
$$= D_{H,t-1}^j (1 + i_{t-1}) + D_{F,t-1}^j (1 + i_{t-1}^*) E_t + M_{H,t-1}^j + W_t N_t^j + PR_t^j - P_t TR_t^j \quad ,$$

$$\int_0^{\nu} P_t^*(h) C_t^{*,j}(h) dh + \int_{\nu}^1 P_t^*(f) C_t^{*,j}(f) dh + \frac{D_{H,t}^{*,j}}{E_t} + D_{F,t}^{*,j} + M_t^{*,j}$$
$$= \frac{D_{H,t-1}^{*,j}}{E_t} (1 + i_{t-1}) + D_{F,t-1}^{*,j} (1 + i_{t-1}^*) + M_{t-1}^{*,j} + W_t^* N_t^{*,j} + PR_t^{*,j} - P_t^* TR_t^{*,j} \quad ,$$

where $D_{H,t}^j$ denotes the government debt issued by the government in country *H* while it held by consumer *j* in country H, $D_{H,t}^{*,j}$ denotes the government debt issued by the government in country *H* while held by consumer *j* in country *F*.

Integrating *j* yields:

$$\int_0^{\nu} P_t(h) C_t(h) dh + \int_{\nu}^1 P_t(f) C_t(f) df + D_{H,t} + D_{F,t} E_t + M_t$$
$$= D_{H,t-1} (1 + i_{t-1}) + D_{F,t-1} (1 + i_{t-1}^*) E_t + M_{t-1} + W_t N_t + PR_t - P_t TR_t \quad , \text{(A-1-2)}$$

$$\int_0^{\nu} P_t^*(h) C_t^*(h) dh + \int_{\nu}^1 P_t^*(f) C_t^*(f) dh + \frac{D_{H,t}^*}{E_t} + D_{F,t}^* + M_t^*$$
$$= \frac{D_{H,t-1}^*}{E_t} (1 + i_{t-1}) + D_{F,t-1}^* (1 + i_{t-1}^*) + M_{t-1}^* + W_t^* N_{F,t} + PR_t^* - P_t^* TR_t^* \quad , \text{(A-1-3)}$$

where super script *j* is suppressed and

$$C_t \equiv \left[\nu^{\frac{1}{\zeta}} C_{H,t}^{\frac{\zeta-1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} C_{F,t}^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} . \text{ (A-1-4)}$$

The definition of the CPI in country *H* is given by:

$P_t \equiv \left[\nu P_{H,t}^{1-\zeta} + (1-\nu) P_{F,t}^{1-\zeta}\right]^{\frac{1}{1-\zeta}}$. (A-1-5)

One of minimization problem for households is given by:

$\max\limits_{C_t(h)} C_{H,t}$,

s.t.

$$C_{H,t} \equiv \left[\left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon}} \int_0^{\nu} C_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{and} \quad X_t - \int_0^{\nu} P_t(h) C_t(h) dh = 0.$$

The Lagrangean is given by

$$L \equiv \left[\left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon}} \int_0^{\nu} C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} + \lambda\left(X_t - \int_0^{\nu} P_t(h) C_t(h) dh\right).$$

The FONC is given by:

$$\begin{aligned}\frac{\partial L}{\partial C_t(h)} &= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1}\left[\int_0^{\nu} C_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} C_t(h)^{\frac{\varepsilon-1}{\varepsilon}-1} - \lambda P_t(h), \\ &= 0\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}\lambda P_t(h) &= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} \left[\int_0^{\nu} C_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{1}{\varepsilon-1}} C_t(h)^{-\frac{1}{\varepsilon}} \\ &= \left[\left(\frac{1}{\nu}\right) \int_0^{\nu} C_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{1}{\varepsilon-1}} C_t(h)^{-\frac{1}{\varepsilon}}\end{aligned}.$$

The definition of $C_{H,t}$ can be rewritten as: $C_{H,t}^{\frac{1}{\varepsilon}} = \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon(\varepsilon-1)}} \left[\int_0^{\nu} C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{1}{\varepsilon-1}}$. Plugging this into the previous expression yields:

$$\lambda P_t(h) = \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} C_{H,t}^{\frac{1}{\varepsilon}} \left(\frac{1}{\nu}\right)^{-\frac{1}{\varepsilon(\varepsilon-1)}} C_t(h)^{-\frac{1}{\varepsilon}}$$

$$= \left(\frac{1}{\nu}\right)^{\frac{\varepsilon-1}{\varepsilon(\varepsilon-1)}} C_{H,t}^{\frac{1}{\varepsilon}} C_t(h)^{-\frac{1}{\varepsilon}}$$, which is available for *h'* as follows:

$$= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon}} C_{H,t}^{\frac{1}{\varepsilon}} C_t(h)^{-\frac{1}{\varepsilon}}$$

$$\lambda P_t(h') = \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon}} C_{H,t}^{\frac{1}{\varepsilon}} C_t(h')^{-\frac{1}{\varepsilon}}.$$

Combining both of them yields:

$$\frac{P_t(h)}{P_t(h')} = \left[\frac{C_t(h)}{C_t(h')}\right]^{-\frac{1}{\varepsilon}}$$, which can be rewritten as:

$$C_t(h) = \left[\frac{P_t(h)}{P_t(h')}\right]^{-\varepsilon} C_t(h').$$

Plugging the previous expression into the definition of $C_{H,t}$ yields:

$$C_{H,t} = \left\{ \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon}} \int_0^{\nu} \left[\left(\frac{P_t(h)}{P_t(h')}\right)^{-\varepsilon} C_t(h') \right]^{\frac{\varepsilon-1}{\varepsilon}} dh \right\}^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} P_t(h')^{\varepsilon-1} C_t(h')^{\frac{\varepsilon-1}{\varepsilon}} dh \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

$$= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} dh P_t(h')^{\varepsilon-1} C_t(h')^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} dh \right]^{\frac{\varepsilon}{\varepsilon-1}} P_t(h')^{\varepsilon} C_t(h')$$

Let define:

$$P_{H,t} \equiv \left[\frac{1}{\nu} \int_0^{\nu} P_t(h)^{1-\varepsilon} dh \right]^{\frac{1}{1-\varepsilon}}.$$

By raising both sides to $-\varepsilon$ th power, we have:

$$P_{H,t}^{-\varepsilon}=\left(\frac{1}{\nu}\right)^{-\frac{\varepsilon}{1-\varepsilon}}\left[\int_0^{\nu}P_t\left(h\right)^{1-\varepsilon}dh\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

Plugging this expression into the previous expression yields:

$$\begin{aligned}C_{H,t}&=\left(\frac{1}{\nu}\right)^{\frac{1}{\varepsilon-1}}P_{H,t}^{-\varepsilon}\left(\frac{1}{\nu}\right)^{-\frac{\varepsilon}{\varepsilon-1}}P_t\left(h'\right)^{\varepsilon}C_t\left(h'\right)\\&=\left(\frac{1}{\nu}\right)^{\frac{1-\varepsilon}{\varepsilon-1}}P_{H,t}^{-\varepsilon}P_t\left(h'\right)^{\varepsilon}C_t\left(h'\right)\\&=\nu P_{H,t}^{-\varepsilon}P_t\left(h'\right)^{\varepsilon}C_t\left(h'\right)\end{aligned}.$$

The previous expression can be applicable any good *h*. Then, we have:

$$C_t\left(h\right)=\frac{1}{\nu}\left(\frac{P_t\left(h\right)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t}. \text{ (A-1-6)}$$

Similarly, we have:

$$C_t\left(f\right)=\frac{1}{1-\nu}\left(\frac{P_t\left(f\right)}{P_{F,t}}\right)^{-\varepsilon}C_{F,t}. \text{ (A-1-7)}$$

Note that we assume $C_{F,t}\equiv\left[\left(\frac{1}{1-\nu}\right)^{\frac{1}{\varepsilon}}\int_{\nu}^{1}C_t\left(f\right)^{\frac{\varepsilon-1}{\varepsilon}}dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and

$$P_{F,t}\equiv\left[\frac{1}{1-\nu}\int_{\nu}^{1}P_t\left(f\right)^{1-\varepsilon}df\right]^{\frac{1}{1-\varepsilon}}.$$

Plugging Eqs.(A-1-6) and (A-1-7) into $\int_0^{\nu}P_t\left(h\right)C_t\left(h\right)dh+\int_{\nu}^{1}P_t\left(f\right)C_t\left(f\right)df$ yields:

$$\int_0^{\nu} P_t(h) C_t(h) dh + \int_{\nu}^{1} P_t(f) C_t(f) df = \int_0^{\nu} P_t(h) \frac{1}{\nu} \left(\frac{P_t(h)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t} dh$$

$$+ \int_{\nu}^{1} P_t(f) \frac{1}{1-\nu} \left(\frac{P_t(f)}{P_{F,t}} \right)^{-\varepsilon} C_{F,t} df$$

$$= \left[\frac{1}{\nu} \int_0^{\nu} P_t(h)^{1-\varepsilon} dh \right] P_{H,t}^{\varepsilon} C_{H,t} \qquad \text{, (A-1-8)}$$

$$+ \left[\frac{1}{1-\nu} \int_{\nu}^{1} P_t(f)^{1-\varepsilon} df \right] P_{F,t}^{\varepsilon} C_{F,t}$$

$$= P_{H,t}^{1-\varepsilon} P_{H,t}^{\varepsilon} C_{H,t} + P_{F,t}^{1-\varepsilon} P_{F,t}^{\varepsilon} C_{F,t}$$

$$= P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$$

where we use $P_{H,t}^{1-\varepsilon} = \frac{1}{\nu} \int_0^{\nu} P_t(h)^{1-\varepsilon} dh$ and $P_{F,t}^{1-\varepsilon} = \frac{1}{1-\nu} \int_{\nu}^{1} P_t(f)^{1-\varepsilon} df$.

Now, we get total demand for goods. Optimization problem is given by:

$\max_{C_{H,t}^j, C_{F,t}^j} C_t^j$,

s.t.

Eq.(A-1-4) and $X_t^j - \left(P_{H,t} C_{H,t}^j + P_{F,t} C_{F,t}^j \right) = 0$.

The Lagrangean is given by

$$L \equiv \left[\nu^{\frac{1}{\zeta}} \left(C_{H,t}^j \right)^{\frac{\zeta-1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} \left(C_{F,t}^j \right)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} + \lambda \left[X_t - \left(P_{H,t} C_{H,t}^j + P_{F,t} C_{F,t}^j \right) \right].$$

The FONCs is given by:

$$
\begin{aligned}
\frac{\partial L}{\partial C_{H,t}^{j}} &= \frac{\gamma}{\gamma-1}\left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}-1}\nu^{\frac{1}{\zeta}}\frac{\zeta-1}{\zeta}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}-1}-\lambda P_{H,t} \\
&= \left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta-(\zeta-1)}{\zeta-1}}\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{-\frac{\zeta-1-\zeta}{\zeta}}-\lambda P_{H,t} \\
&= \left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{1}{\zeta-1}}\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{-\frac{1}{\zeta}}-\lambda P_{H,t} \qquad , \\
&= \left(C_{t}^{j}\right)^{\frac{1}{\zeta}}\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{-\frac{1}{\zeta}}-\lambda P_{H,t} \\
&= \nu^{\frac{1}{\zeta}}\left(\frac{C_{t}^{j}}{C_{H,t}^{j}}\right)^{\frac{1}{\zeta}}-\lambda P_{H,t} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial C_{F,t}^{j}} &= \frac{\zeta}{\zeta-1}\left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}-1}(1-\nu)^{\frac{1}{\zeta}}\frac{\zeta-1}{\zeta}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}-1}-\lambda P_{F,t} \\
&= \left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta-(\zeta-1)}{\zeta-1}}(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{-\frac{\zeta-1-\zeta}{\zeta}}-\lambda P_{F,t} \\
&= \left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{1}{\zeta-1}}(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{-\frac{1}{\zeta}}-\lambda P_{F,t} \\
&= \left(C_{t}^{j}\right)^{\frac{1}{\zeta}}(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{-\frac{1}{\zeta}}-\lambda P_{F,t} \\
&= (1-\nu)^{\frac{1}{\zeta}}\left(\frac{C_{t}^{j}}{C_{F,t}^{j}}\right)^{\frac{1}{\zeta}}-\lambda P_{F,t} \\
&= 0
\end{aligned}
$$

where we use $\left(C_{t}^{j}\right)^{\frac{1}{\zeta}}=\left[\nu^{\frac{1}{\zeta}}\left(C_{H,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}+(1-\nu)^{\frac{1}{\zeta}}\left(C_{F,t}^{j}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{1}{\zeta-1}}$.

These previous expressions can be rewritten as:

$$
\nu^{\frac{1}{\zeta}}\left(\frac{C_{t}^{j}}{C_{H,t}^{j}}\right)^{\frac{1}{\zeta}}=\lambda P_{H,t},
$$

$$(1-\nu)^{\frac{1}{\zeta}}\left(\frac{C_t^j}{C_{F,t}^j}\right)^{\frac{1}{\zeta}}=\lambda P_{F,t}\ .$$

Combining these expression yields:

$$\left(\frac{1-\nu}{\nu}\right)^{\frac{1}{\zeta}}\left(\frac{C_{H,t}^j}{C_{F,t}^j}\right)^{\frac{1}{\zeta}}=S_t\ ,\ \text{(A-1-9)}$$

with $S_t \equiv P_{F,t}/P_{H,t}$.

Eq.(A-1-5) can be rewritten as:

$$P_t^{1-\zeta}=\nu P_{H,t}^{1-\zeta}+(1-\nu)P_{F,t}^{1-\zeta}\ .\ \text{(A-1-10)}$$

Dividing both sides of the Eq.(A-1-10) by $P_{H,t}^{1-\zeta}$ yields:

$$\left(\frac{P_t}{P_{H,t}}\right)^{1-\zeta}=\nu+(1-\nu)S_t^{1-\zeta}\ ,$$

which can be rewritten as:

$$S_t^{1-\zeta}=\frac{1}{1-\nu}\left(\frac{P_t}{P_{H,t}}\right)^{1-\zeta}-\nu(1-\nu)^{-1}\ .\ \text{(A-1-11)}$$

Dividing both sides of the Eq.(A-1-10) by $P_{F,t}^{1-\zeta}$ yields:

$$\left(\frac{P_{F,t}}{P_t}\right)^{-(1-\zeta)}=\nu S_t^{\zeta-1}+(1-\nu)\ ,$$

which can be rewritten as:

$$S_t^{\zeta-1}=\frac{1}{\nu}\left(\frac{P_{F,t}}{P_t}\right)^{-(1-\zeta)}-\frac{1-\nu}{\nu}\ .\ \text{(A-1-12)}$$

Plugging Eq.(A-1-9) $C_{F,t}^j=\left(\frac{1-\nu}{\nu}\right)C_{H,t}^j S_t^{-\zeta}$ into Eq.(A-1-4) yields:

$$C_t^j = \left\{ \nu^{\frac{1}{\zeta}} \left(C_{H,t}^j \right)^{\frac{\zeta-1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} \left[S_t^{-\zeta} \left(\frac{1-\nu}{\nu} \right) C_{H,t}^j \right]^{\frac{\zeta-1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}}$$

$$= \left\{ \nu^{\frac{1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} \left[S_t^{1-\zeta} \left(\frac{1-\nu}{\nu} \right)^{\frac{\zeta-1}{\zeta}} \right] \left(C_{H,t}^j \right)^{\frac{\zeta-1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}}$$

$$= \left[\nu^{\frac{1}{\zeta}} + (1-\nu) S_t^{1-\zeta} \nu^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} C_{H,t}^j \quad .$$

$$= \left[\left(1 + \frac{1-\nu}{\nu} S_t^{1-\zeta} \right) \nu^{\frac{1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} C_{H,t}^j$$

$$= \nu^{\frac{1}{\zeta-1}} \left(1 + \frac{1-\nu}{\nu} S_t^{1-\zeta} \right)^{\frac{\zeta}{\zeta-1}} C_{H,t}^j$$

Plugging Eq.(A-1-11) into the previous expression yields:

$$C_t^j = \nu^{\frac{1}{\zeta-1}} \left[1 + \frac{1}{\nu} \left(\frac{P_t}{P_{H,t}} \right)^{1-\zeta} - 1 \right]^{\frac{\zeta}{\zeta-1}} C_{H,t}^j$$

$$= \nu^{\frac{1}{\zeta-1}} \left[\frac{1}{\nu} \left(\frac{P_t}{P_{H,t}} \right)^{1-\gamma} \right]^{\frac{\zeta}{\zeta-1}} C_{H,t}^j \quad ,$$

$$= \nu^{\frac{1}{\zeta-1}} \nu^{-\frac{\zeta}{\zeta-1}} \left(\frac{P_t}{P_{H,t}} \right)^{-\zeta} C_{H,t}^j$$

$$= \frac{1}{\nu} \left(\frac{P_t}{P_{H,t}} \right)^{-\zeta} C_{H,t}^j$$

which can be rewritten as:

$$C_{H,t}^j = \nu \left(\frac{P_{H,t}}{P_t} \right)^{-\zeta} C_t^j \, .$$

Integrating the previous expression *j* yields:

$$C_{H,t} = \nu \left(\frac{P_{H,t}}{P_t} \right)^{-\zeta} C_t^W , \text{ (A-1-13)}$$

with $C_t^W \equiv \int_0^1 C_t^j dj$.

Plugging Eq.(A-1-9) $C_{H,t}^j = \left(\frac{\nu}{1-\nu}\right) S_t^{\zeta} C_{F,t}^j$ into Eq.(A-1-4) yields:

$$
\begin{aligned}
C_t^j &= \left\{ \nu^{\frac{1}{\zeta}} \left[\left(\frac{\nu}{1-\nu}\right) S_t^{\zeta} C_{F,t}^j \right]^{\frac{\zeta-1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} \left(C_{F,t}^j\right)^{\frac{\zeta-1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}} \\
&= \left\{ \nu^{\frac{1}{\zeta}} \left(\frac{\nu}{1-\nu}\right)^{\frac{\zeta-1}{\zeta}} \left(S_t^{\zeta}\right)^{\frac{\zeta-1}{\zeta}} \left(C_{F,t}^j\right)^{\frac{\zeta-1}{\zeta}} + (1-\nu)^{\frac{1}{\zeta}} \left(C_{F,t}^j\right)^{\frac{\zeta-1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}} \\
&= \left\{ \left[\nu^{\frac{1}{\zeta}} \nu^{\frac{\zeta-1}{\zeta}} (1-\nu)^{-\frac{\zeta-1}{\zeta}} S_t^{\zeta-1} + (1-\nu)^{\frac{1}{\zeta}} \right] \left(C_{F,t}^j\right)^{\frac{\zeta-1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}} \\
&= \left\{ \left[\nu (1-\nu)^{-1} S_t^{\zeta-1} + 1 \right] (1-\nu)^{\frac{1}{\zeta}} \right\}^{\frac{\zeta}{\zeta-1}} C_{F,t}^j \\
&= \left[\nu (1-\nu)^{-1} S_t^{\zeta-1} + 1 \right]^{\frac{\zeta}{\zeta-1}} (1-\nu)^{\frac{1}{\zeta-1}} C_{F,t}^j
\end{aligned} .
$$

Plugging Eq.(A-1-12) into the previous expression yields:

$$
\begin{aligned}
C_t^j &= \left[\nu (1-\nu)^{-1} \left[\frac{1}{\nu} \left(\frac{P_{F,t}}{P_t}\right)^{-(1-\zeta)} - \frac{1-\nu}{\nu} \right] + 1 \right]^{\frac{\zeta}{\zeta-1}} (1-\nu)^{\frac{1}{\zeta-1}} C_{F,t}^j \\
&= \left[\left[\frac{1}{1-\nu} \left(\frac{P_{F,t}}{P_t}\right)^{-(1-\zeta)} - 1 \right] + 1 \right]^{\frac{\zeta}{\zeta-1}} (1-\nu)^{\frac{1}{\zeta-1}} C_{F,t}^j \\
&= \left(\frac{1}{1-\nu}\right)^{\frac{\zeta}{\zeta-1}} \left(\frac{P_{F,t}}{P_t}\right)^{\zeta} (1-\nu)^{\frac{1}{\zeta-1}} C_{F,t}^j \\
&= \frac{1}{1-\nu} \left(\frac{P_{F,t}}{P_t}\right)^{\zeta} C_{F,t}^j
\end{aligned} ,
$$

which can be rewritten as:

$$
C_{F,t}^j = (1-\nu) \left(\frac{P_{F,t}}{P_t}\right)^{-\zeta} C_t^j .
$$

Integrating the previous expression *j* yields:

$$C_{F,t}=(1-\nu)\left(\frac{P_{F,t}}{P_t}\right)^{-\zeta}C_t^{w}\text{ . (A-1-14)}$$

Plugging Eqs.(A-1-13) and (A-1-14) into $P_{H,t}C_{H,t}+P_{F,t}C_{F,t}$ yields:

$$\begin{aligned}
P_{H,t}C_{H,t}+P_{F,t}C_{F,t}&=P_{H,t}\nu\left(\frac{P_{H,t}}{P_t}\right)^{-\zeta}C_t+P_{F,t}(1-\nu)\left(\frac{P_{F,t}}{P_t}\right)^{-\zeta}C_t\\
&=\left[\nu P_{H,t}^{1-\zeta}\frac{P_t^{\zeta}}{P_t}+(1-\nu)P_{F,t}^{1-\zeta}\frac{P_t^{\zeta}}{P_t}\right]P_tC_t\\
&=\left[\nu P_{H,t}^{1-\zeta}P_t^{\zeta-1}+(1-\nu)P_{F,t}^{1-\zeta}P_t^{\zeta-1}\right]P_tC_t\\
&=\left[\nu P_{H,t}^{1-\zeta}P_t^{-(1-\zeta)}+(1-\nu)P_{F,t}^{1-\zeta}P_t^{-(1-\zeta)}\right]P_tC_t\\
&=\left[\nu\left(\frac{P_{H,t}}{P_t}\right)^{1-\zeta}+(1-\nu)\left(\frac{P_{F,t}}{P_t}\right)^{1-\zeta}\right]P_tC_t\\
&=\left[\frac{\nu}{\nu+(1-\nu)S_t^{1-\zeta}}+\frac{1-\nu}{\nu S_t^{\zeta-1}+(1-\nu)}\right]P_tC_t\\
&=\left\{\frac{\nu\left[\nu S_t^{\zeta-1}+(1-\nu)\right]+(1-\nu)\left[\nu+(1-\nu)S_t^{1-\zeta}\right]}{\left[\nu+(1-\nu)S_t^{1-\zeta}\right]\left[\nu S_t^{\zeta-1}+(1-\nu)\right]}\right\}P_tC_t\\
&=\left\{\frac{\nu\left[\nu S_t^{\zeta-1}+(1-\nu)\right]+(1-\nu)\left[\nu+(1-\nu)S_t^{1-\zeta}\right]}{\nu^2S_t^{\zeta-1}+\nu(1-\nu)+\nu(1-\nu)S_t^{1-\zeta}S_t^{\zeta-1}+(1-\nu)^2S_t^{1-\zeta}}\right\}P_tC_t\\
&=\left\{\frac{\nu\left[\nu S_t^{\zeta-1}+(1-\nu)\right]+(1-\nu)\left[\nu+(1-\nu)S_t^{1-\zeta}\right]}{\nu\left[S_t^{\zeta-1}+(1-\nu)\right]+(1-\nu)\left[\nu+(1-\nu)S_t^{1-\zeta}\right]}\right\}P_tC_t\\
&=P_tC_t
\end{aligned}\text{ . (A-1-15)}$$

Plugging Eqs(A-1-8) and (A-1-15) into Eqs.(A-1-2) and (A-1-3) yields:

$$\begin{aligned}
&P_tC_t+D_{H,t}+D_{F,t}E_t+M_t\\
&=D_{H,t-1}(1+i_{t-1})+D_{F,t-1}(1+i_{t-1}^{*})E_t+M_{t-1}+W_tN_t+PR_t-P_tTR_t
\end{aligned}\text{ , (A-1-16)}$$

$$\begin{aligned}
&P_t^{*}C_t^{*}+\frac{D_{H,t}^{*}}{E_t}+D_{F,t}^{*}+M_t^{*}\\
&=\frac{D_{H,t-1}^{*}}{E_t}(1+i_{t-1})+D_{F,t-1}^{*}(1+i_{t-1}^{*})+M_{t-1}^{*}+W_t^{*}N_t^{*}+PR_t^{*}-P_t^{*}TR_t^{*}
\end{aligned}\text{ , (A-1-17)}$$

where $B_t=D_{H,t}+\dfrac{D_{H,t}^{*}}{E_t}$ and $B_t^{*}=E_tB_{F,t}+B_{F,t}^{*}$.

Dividing both sides of Eq.(A-1-16) and (A-1-1-17) by P_t and P_t^* yields:

$$C_t + \mathrm{D}_{H,t} + \frac{E_t P_t^*}{P_t}\mathrm{D}_{F,t} + L_t = \frac{P_{t-1}}{P_t}\mathrm{D}_{H,t-1}\left(1+i_{t-1}\right) + \frac{E_t P_t^*}{P_t}\frac{P_{t-1}^*}{P_t^*}\mathrm{D}_{F,t-1}\left(1+i_{t-1}^*\right) + \frac{P_{t-1}}{P_t}L_{t-1} + \frac{W_t}{P_t}N_t + \frac{PR_t}{P_t} - TR_t ,$$

$$C_t^* + \frac{P_t}{P_t^* E_t}\mathrm{D}_{H,t}^* + \mathrm{D}_{F,t}^* + L_t^* = \frac{P_t}{P_t^* E_t}\frac{P_{t-1}}{P_t}\mathrm{D}_{H,t-1}^*\left(1+i_{t-1}\right) + \frac{P_{t-1}^*}{P_t^*}\mathrm{D}_{F,t-1}^*\left(1+i_{t-1}^*\right) + \frac{P_{t-1}^*}{P_t^*}L_{t-1}^* + \frac{W_t^*}{P_t^*}N_t^* + \frac{PR_t^*}{P_t^*} - TR_t^* ,$$

with $\mathrm{D}_{H,t} \equiv \frac{D_{H,t}}{P_t}$, $\mathrm{D}_{F,t} \equiv \frac{D_{F,t}}{P_t^*}$, $\mathrm{D}_{H,t}^* \equiv \frac{D_{H,t}^*}{P_t}$, $\mathrm{D}_{F,t}^* \equiv \frac{D_{F,t}^*}{P_t^*}$, $L_t \equiv \frac{M_t}{P_t}$ and $L_t^* \equiv \frac{M_t^*}{P_t^*}$.

Previous expressions can be rewritten as:

$$C_t + \mathrm{D}_{H,t} + \frac{E_t P_t^*}{P_t}\mathrm{D}_{F,t} + L_t = \frac{P_{t-1}}{P_t}\mathrm{D}_{H,t-1}\left(1+i_{t-1}\right) + \frac{P_{t-1}}{P_t}\frac{E_t}{E_{t-1}}\frac{E_{t-1}P_{t-1}^*}{P_{t-1}}\mathrm{D}_{F,t-1}\left(1+i_{t-1}^*\right) + \frac{P_{t-1}}{P_t}L_{t-1} + \frac{W_t}{P_t}N_t + \frac{PR_t}{P_t} - TR_{H,t} ,$$

$$C_t^* + \frac{P_t}{P_t^* E_t}\mathrm{D}_{H,t}^* + \mathrm{D}_{F,t}^* + L_t^* = \frac{P_{t-1}^*}{P_t^*}\frac{E_{t-1}}{E_t}\frac{P_{t-1}}{E_{t-1}P_{t-1}^*}\mathrm{D}_{H,t-1}^*\left(1+i_{t-1}\right) + \frac{P_{t-1}^*}{P_t^*}\mathrm{D}_{F,t-1}^*\left(1+i_{t-1}^*\right) + \frac{P_{t-1}^*}{P_t^*}L_{t-1}^* + \frac{W_{F,t}^*}{P_t^*}N_t^* + \frac{PR_t^*}{P_t^*} - TR_t^* .$$

Let define $Q_t \equiv \frac{E_t P_t^*}{P_t}$. Then, the previous expression can be rewritten as:

$$C_t + \mathrm{D}_{H,t} + Q_t\mathrm{D}_{F,t} + L_t = \left[\mathrm{D}_{H,t-1}\left(1+i_{t-1}\right) + \frac{E_t}{E_{t-1}}Q_{t-1}\mathrm{D}_{F,t-1}\left(1+i_{t-1}^*\right) + L_{t-1}\right]\Pi_t^{-1} + \frac{W_t}{P_t}N_t + \frac{PR_t}{P_t} - TR_t ,$$

$$C_t^* + Q_t^{-1}\mathrm{D}_{H,t}^* + \mathrm{D}_{F,t}^* + L_t^* = \left[\frac{E_{t-1}}{E_t}Q_{t-1}^{-1}\mathrm{D}_{H,t-1}^*\left(1+i_{t-1}\right) + \mathrm{D}_{F,t-1}^*\left(1+i_{t-1}^*\right) + L_{t-1}^*\right]\left(\Pi_{t-1}^*\right)^{-1} + \frac{W_t^*}{P_t^*}N_t^* + \frac{PR_t^*}{P_t^*} - TR_t^* .$$

Plugging $1+i_t=\left(1+i_t^*\right)\left(\frac{E_{t+1}}{E_t}\right)$ or $1+i_t^*=\left(1+i_t\right)\left(\frac{E_t}{E_{t+1}}\right)$ into the previous expressions yields:

$$C_t+\mathrm{D}_{H,t}+Q_t\mathrm{D}_{F,t}+L_t=\left[\mathrm{D}_{H,t-1}\left(1+i_{t-1}\right)+Q_{t-1}\mathrm{D}_{F,t-1}\left(1+i_{t-1}\right)+L_{t-1}\right]\Pi_t^{-1} \\ +\frac{W_t}{P_t}N_t+\frac{PR_t}{P_t}-TR_t ,$$

$$C_t^*+Q_t^{-1}\mathrm{D}_{H,t}^*+\mathrm{D}_{F,t}^*+L_t^*=\left[Q_{t-1}^{-1}\mathrm{D}_{H,t-1}^*\left(1+i_{t-1}^*\right)+\mathrm{D}_{F,t-1}^*\left(1+i_{t-1}^*\right)+L_{t-1}^*\right]\left(\Pi_{t-1}^*\right)^{-1} \\ +\frac{W_t^*}{P_t^*}N_t^*+\frac{PR_t^*}{P_t^*}-TR_t^* .$$

Further, the previous expression can be rewritten as:

$$C_t+\frac{1}{1+i_t}\left[\left(1+i_t\right)\mathrm{D}_{H,t}+\left(1+i_t\right)Q_t\mathrm{D}_{F,t}+L_t\right]+\left(1-\frac{1}{1+i_t}\right)L_t \\ =\left[\left(1+i_{t-1}\right)\mathrm{D}_{H,t-1}+\left(1+i_{t-1}\right)Q_{t-1}\mathrm{D}_{F,t-1}+L_{t-1}\right]\Pi_t^{-1}+\frac{W_t}{P_t}N_t+\frac{PR_t}{P_t}-TR_t ,$$

$$C_t^*+\frac{1}{1+i_t^*}\left[Q_t^{-1}\left(1+i_t^*\right)\mathrm{D}_{H,t}^*+\left(1+i_t^*\right)\mathrm{D}_{F,t}^*+L_t^*\right]+\left(1-\frac{1}{1+i_t^*}\right)L_t^* \\ =\left[Q_{t-1}^{-1}\mathrm{D}_{H,t-1}^*\left(1+i_{t-1}^*\right)+\mathrm{D}_{F,t-1}^*\left(1+i_{t-1}^*\right)+L_{t-1}^*\right]\left(\Pi_t^*\right)^{-1}+\frac{W_t^*}{P_t^*}N_t^*+\frac{PR_t^*}{P_t^*}-TR_t^* .$$

Let define $\mathrm{A}_t\equiv\left[\left(1+i_{t-1}\right)\mathrm{D}_{H,t-1}+\left(1+i_{t-1}\right)Q_{t-1}\mathrm{D}_{F,t-1}+L_{t-1}\right]\Pi_t^{-1}$ and $\mathrm{A}_t^*\equiv\left[\left(1+i_{t-1}^*\right)Q_{t-1}^{-1}\mathrm{D}_{H,t-1}^*+\left(1+i_{t-1}^*\right)\mathrm{D}_{F,t-1}^*+L_{t-1}^*\right]\left(\Pi_t^*\right)^{-1}$. Then, the previous expression can be rewritten as:

$$C_t+\frac{1}{1+i_t}\Pi_{t+1}\mathrm{A}_{t+1}+\frac{i_t}{1+i_t}L_t=\mathrm{A}_t+\frac{W_t}{P_t}N_t+\frac{PR_t}{P_t}-TR_t, \quad \text{(A-1-18)}$$

$$C_t^*+\frac{1}{1+i_t^*}\Pi_{t+1}^*\mathrm{A}_{t+1}^*+\frac{i_t^*}{1+i_t^*}L_t^*=\mathrm{A}_t^*+\frac{W_t^*}{P_t^*}N_t^*+\frac{PR_t^*}{P_t^*}-TR_t^* . \quad \text{(A-1-19)}$$

Households' optimization problem is given by:

$$\max_{C_t,C_{t+1},N_t,\mathrm{A}_{t+1},L_t}\sum_{t=0}^{\infty}\beta^t U_t , \quad \max_{C_t^*,C_{t+1}^*,N_t^*,\mathrm{A}_{t+1}^*,L_t^*}\sum_{t=0}^{\infty}\beta^t U_t^*$$

s.t.

$U_t \equiv \left[U\left(C_t, L_t\right) - V\left(N_t\right)\right] Z_t$, $U_t^* \equiv \left[U\left(C_t^*, L_t^*\right) - V\left(N_t^*\right)\right] Z_t^*$

and Eqs.(A-1-18) and (A-1-19).

The Lagrangean is given by:

$$
\begin{aligned}
L \equiv\ & \beta^t \left[U\left(C_t, L_t\right) - V\left(N_t\right)\right] Z_t + \beta^{t+1} \left[U\left(C_{t+1}, L_{t+1}\right) - V\left(N_{t+1}\right)\right] Z_{t+1} + \cdots \\
& + \lambda_t \beta^t \left(\mathrm{A}_t + \frac{W_t}{P_t} N_t + \frac{PR_t}{P_t} - TR_t - C_t - \frac{1}{1+i_t} \Pi_{t+1} \mathrm{A}_{t+1} - \frac{i_t}{1+i_t} L_t \right) \\
& + \lambda_{t+1} \beta^{t+1} \left(\mathrm{A}_{t+1} + \frac{W_{t+1}}{P_{t+1}} N_{t+1} + \frac{PR_{t+1}}{P_{t+1}} - TR_{t+1} - C_{t+1} - \frac{1}{1+i_{t+1}} \Pi_{t+2} \mathrm{A}_{t+2} - \frac{i_{t+1}}{1+i_{t+1}} L_{t+1} \right) \\
& + \cdots
\end{aligned}
,
$$

$$
\begin{aligned}
L^* \equiv\ & \beta^t \left[U\left(C_t^*, L_t^*\right) - V\left(N_t^*\right)\right] Z_t^* + \beta^{t+1} \left[U\left(C_{t+1}^*, L_{t+1}^*\right) - V\left(N_{t+1}^*\right)\right] Z_{t+1}^* + \cdots \\
& + \lambda_t^* \beta^t \left(\mathrm{A}_t^* + \frac{W_t^*}{P_t^*} N_t^* + \frac{PR_t^*}{P_t^*} - TR_t^* - C_t^* - \frac{1}{1+i_t^*} \Pi_{t+1}^* \mathrm{A}_{t+1}^* - \frac{i_t^*}{1+i_t^*} L_t^* \right) \\
& + \lambda_{t+1}^* \beta^{t+1} \left(\mathrm{A}_{t+1}^* + \frac{W_{t+1}^*}{P_{t+1}^*} N_{t+1}^* + \frac{PR_{t+1}^*}{P_{t+1}^*} - TR_{t+1}^* - C_{t+1}^* - \frac{1}{1+i_t^*} \Pi_{t+2}^* \mathrm{A}_{t+2}^* - \frac{i_{t+1}^*}{1+i_{t+1}^*} L_{t+1}^* \right) \\
& + \cdots
\end{aligned}
.
$$

FONCs are given by:

$$\frac{\partial L}{\partial C_t} = \beta^t U_{c,t} Z_t - \beta^t \lambda_t = 0,$$

$$\frac{\partial L}{\partial C_{t+1}} = \beta^{t+1} U_{c,t+1} Z_{t+1} - \beta^{t+1} \lambda_{t+1} = 0,$$

$$\frac{\partial L}{\partial N_t} = \beta^t \left(-V_{n,t}\right) Z_t + \lambda_t \beta^t \frac{W_t}{P_t} = 0,$$

$$\frac{\partial L}{\partial \mathrm{A}_{t+1}} = -\lambda_t \beta^t \frac{1}{1+i_t} \Pi_{t+1} + \lambda_{t+1} \beta^{t+1} = 0,$$

$$\frac{\partial L}{\partial L_t} = \beta^t U_{l,t} Z_t - \beta^t \lambda_t \left(\frac{i_t}{1+i_t} \right) = 0,$$

$$\frac{\partial L^*}{\partial C_t^*} = \beta^t U_{c,t}^* Z_t^* - \beta^t \lambda_t^* = 0,$$

$$\frac{\partial L^*}{\partial C_{t+1}^*} = \beta^{t+1} U_{c,t+1}^* Z_{t+1}^* - \beta^{t+1} \lambda_{t+1}^* = 0,$$

$$\frac{\partial L^*}{\partial N_t^*} = \beta^t \left(-V_{n^*,t}\right) Z_t^* + \lambda_t^* \beta^t \frac{W_t^*}{P_t^*} = 0,$$

$$\frac{\partial L^*}{\partial \mathrm{A}^*_{t+1}} = -\lambda^*_t \beta^t \frac{1}{1+i^*_t}\Pi^*_{t+1} + \lambda^*_{t+1}\beta^{t+1} = 0\,,$$

$$\frac{\partial L^*}{\partial L^*_t} = \beta^t U_{L^*,t} Z^*_t - \beta^t \lambda^*_t \left(\frac{i^*_t}{1+i^*_t}\right) = 0\,,$$

which can be rewritten as:

$\lambda_t = U_{c,t} Z_t$, (A-1-20)

$\lambda_{t+1} = U_{c,t+1} Z_{t+1}$, (A-1-21)

$\lambda_t = \left(\frac{W_t}{P_t}\right)^{-1} V_{n,t} Z_t$, (A-1-22)

$\lambda_t = \beta \lambda_{t+1} \Pi^{-1}_{t+1} \left(1+i_t\right)$, (A-1-23)

$\lambda_t = U_{l,t} \left(\frac{i_t}{1+i_t}\right)^{-1} Z_t$, (A-1-24)

$\lambda^*_t = U^*_{c,t} Z^*_t$, (A-1-25)

$\lambda^*_{t+1} = U^*_{c,t+1} Z^*_{t+1}$, (A-1-26)

$\lambda^*_t = V_{n^*,t} \left(\frac{W^*_t}{P^*_t}\right)^{-1} Z^*_t$, (A-1-27)

$\lambda^*_t = \beta \lambda^*_{t+1} \left(\Pi^*_{t+1}\right)^{-1} \left(1+i^*_t\right)$, (A-1-28)

$\lambda^*_t = U_{l^*,t} \left(\frac{i^*_t}{1+i^*_t}\right)^{-1} Z^*_t$, (A-1-29)

where λ_t and λ^*_t es Lagrange multipliers

Combining Eqs.(A-1-20), (A-1-21) and (A-1-23) yields:

$U_{c,t} = \beta\left(1+i_t\right)\Pi^{-1}_{t+1} U_{c,t+1} \left(\frac{Z_{t+1}}{Z_t}\right)$. (A-1-30)

Combining Eqs.(A-1-25) (A-1-26)and (A-1-28)yields:

$$U_{c,t}^{*}=\beta\left(1+i_{t}^{*}\right)\left(\Pi_{t+1}^{*}\right)^{-1}U_{c,t+1}^{*}\left(\frac{Z_{t+1}^{*}}{Z_{t}^{*}}\right). \text{ (A-1-31)}$$

Combing Eqs.(A-1-20) and (A-1-22) yields:

$$\frac{W_{t}}{P_{t}}=\frac{V_{n,t}}{U_{c,t}}. \text{ (A-1-32)}$$

Combing Eqs.(A-1-25) and (A-1-27) yields:

$$\frac{W_{t}^{*}}{P_{t}^{*}}=\frac{V_{n^{*},t}}{U_{c,t}^{*}}. \text{ (A-1-33)}$$

Combing Eqs.(A-1-20) and (A-1-24) yields:

$$\frac{U_{l,t}}{U_{c,t}}=\frac{i_{t}}{1+i_{t}}. \text{ (A-1-34)}$$

Combing Eqs.(A-1-20) and (A-1-24) yields:

$$\frac{U_{l^{*},t}}{U_{c,t}^{*}}=\frac{i_{t}^{*}}{1+i_{t}^{*}}. \text{ (A-1-35)}$$

Combing Eqs.(A-1-30) and (A-1-31) yields:

$$\frac{\lambda_{t}}{\lambda_{t}^{*}}=\frac{1+i_{t}}{1+i_{t}^{*}}\frac{\Pi_{t+1}^{*}}{\Pi_{t+1}}\frac{\lambda_{t+1}}{\lambda_{t+1}^{*}}. \text{ (A-1-36)}$$

1.2 LOOP and PPP

We assume the law of one price as follows:

$$P_{t}(h)=E_{t}P_{t}^{*}(h),$$

$$P_{t}(f)=E_{t}P_{t}^{*}(f).$$

Plugging the previous expressions into $P_{H,t}\equiv\left[\frac{1}{\nu}\int_{0}^{\nu}P_{t}(h)^{1-\varepsilon}dh\right]^{\frac{1}{1-\varepsilon}}$ and $P_{F,t}\equiv\left[\frac{1}{1-\nu}\int_{\nu}^{1}P_{t}(f)^{1-\varepsilon}df\right]^{\frac{1}{1-\varepsilon}}$ yields:

$$P_{H,t} = \left\{ \frac{1}{\nu} \int_0^{\nu} \left[E_t P_t^*(h) \right]^{1-\varepsilon} dh \right\}^{\frac{1}{1-\varepsilon}}$$
$$= \left[\frac{1}{\nu} E_t^{1-\varepsilon} \int_0^{\nu} P_t^*(h)^{1-\varepsilon} dh \right]^{\frac{1}{1-\varepsilon}} ,$$
$$= E_t \left[\frac{1}{\nu} \int_0^{\nu} P_t^*(h)^{1-\varepsilon} dh \right]^{\frac{1}{1-\varepsilon}}$$
$$= E_t P_{H,t}^*$$

$$P_{F,t} = \left\{ \frac{1}{1-\nu} \int_{\nu}^{1} \left[E_t P_t^*(f) \right]^{1-\varepsilon} df \right\}^{\frac{1}{1-\varepsilon}}$$
$$= \left[\frac{1}{1-\nu} E_t^{1-\varepsilon} \int_{\nu}^{1} P_t^*(f)^{1-\varepsilon} df \right]^{\frac{1}{1-\varepsilon}} .$$
$$= E_t \left[\frac{1}{1-\nu} \int_{\nu}^{1} P_t^*(f)^{1-\varepsilon} df \right]^{\frac{1}{1-\varepsilon}}$$
$$= E_t P_{F,t}^*$$

Plugging $P_{H,t} = E_t P_{H,t}^*$ and $P_{F,t} = E_t P_{F,t}^*$ into Eq.(A-1-5) $P_t \equiv \left[\nu P_{H,t}^{1-\zeta} + (1-\nu) P_{F,t}^{1-\zeta} \right]^{\frac{1}{1-\zeta}}$ yields:

$$P_t = \left[\nu \left(E_t P_{H,t}^* \right)^{1-\zeta} + (1-\nu) \left(E_t P_{F,t}^* \right)^{1-\zeta} \right]^{\frac{1}{1-\zeta}}$$
$$= \left[\nu E_t^{1-\zeta} \left(P_{H,t}^* \right)^{1-\zeta} + (1-\nu) E_t^{1-\zeta} \left(P_{F,t}^* \right)^{1-\zeta} \right]^{\frac{1}{1-\zeta}}$$
$$= \left\{ E_t^{1-\gamma} \left[\nu \left(P_{H,t}^* \right)^{1-\zeta} + (1-\nu) \left(P_{F,t}^* \right)^{1-\zeta} \right] \right\}^{\frac{1}{1-\gamma}} .$$
$$= E_t \left[\nu \left(P_{H,t}^* \right)^{1-\zeta} + (1-\nu) \left(P_{F,t}^* \right)^{1-\zeta} \right]^{\frac{1}{1-\zeta}}$$
$$= E_t P_t^*$$

That is, given the LOOP and the structure of preferences, purchasing power parity (PPP) i.e.,

$P_t = E_t P_t^*$,

holds. Plugging the PPP into the definition of the real exchange rate:
$Q_t = 1$. (A-1-37)
Plugging the PPP into Eq.(A-1-36) yields:

$$\left(1+i_t\right)\frac{\lambda_{t+1}}{\lambda_t}=\left(1+i_t^*\right)\frac{E_{t+1}}{E_t}\frac{\lambda_{t+1}^*}{\lambda_t^*},$$

which is applicable even in period -1 as follows:

$$\frac{1+i_{t-1}}{1+i_{t-1}^*}\frac{\lambda_0}{\lambda_{-1}}=\frac{E_0}{E_{-1}}\frac{\lambda_0^*}{\lambda_{-1}^*}.$$

Let assume complete financial market. Then:

$$\lambda_0=\lambda_0^*,$$

where we assume $\frac{\lambda_{-1}}{\lambda_{-1}^*}=1$.

Previous expression is applicable for all *t* so that:

$\lambda_t=\lambda_t^*$. (A-1-38)

Then, Eq.(A-1-36) can be rewritten as:

$$1+i_t=\left(1+i_t^*\right)\frac{E_{t+1}}{E_t},$$

which is the UIP.

1.4 Domestic Producers

Production function is given by:

$Y_t=A_tN_t^{1-\alpha}$, (A-1-39)

$Y_t^*=A_t^*\left(N_t^*\right)^{1-\alpha}$. (A-1-40)

Let define $Y_t\equiv\left[\left(\frac{1}{\nu}\right)\int_0^{\nu}Y_t\left(h\right)^{\frac{\varepsilon-1}{\varepsilon}}dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $Y_t^*\equiv\left[\left(\frac{1}{1-\nu}\right)\int_{\nu}^{1}Y_t\left(f\right)^{\frac{\varepsilon-1}{\varepsilon}}dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$.

Maximization problem for producers is given by:

$$\max_{Y_t(h)}Y_t,$$

s.t.

$Y_t\equiv\left[\left(\frac{1}{\nu}\right)\int_0^{\nu}Y_t\left(h\right)^{\frac{\varepsilon-1}{\varepsilon}}dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $X_t-\int_0^{\nu}P_t\left(h\right)Y_t\left(h\right)dh=0$.

The Lagrangean is given by

$$L \equiv \left[\left(\frac{1}{\nu} \right) \int_0^{\nu} Y_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} + \lambda \left(X_t - \int_0^{\nu} P_t(h) Y_t(h) dh \right).$$

The FONC is given by:

$$\begin{aligned} \frac{\partial L}{\partial Y_t(h)} &= \left(\frac{1}{\nu} \right)^{\frac{\varepsilon}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} \left[\int_0^{\nu} Y_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh \right]^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} Y_t(h)^{\frac{\varepsilon-1}{\varepsilon}-1} - \lambda P_t(h), \\ &= 0 \end{aligned}$$

which can be rewritten as:

$$\lambda P_t(h) = \left(\frac{1}{\nu} \right)^{\frac{\varepsilon}{\varepsilon-1}} \left[\int_0^{\nu} Y_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh \right]^{\frac{1}{\varepsilon-1}} Y_t(h)^{-\frac{1}{\varepsilon}}.$$

The definition of $Y_{H,t}$ can be rewritten as: $Y_t^{\frac{1}{\varepsilon}} \left(\frac{1}{\nu} \right)^{-\frac{1}{\varepsilon-1}} = \left[\int_0^{\nu} Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{1}{\varepsilon-1}}$. Plugging this into the previous expression yields:

$$\begin{aligned} \lambda P_t(h) &= \left(\frac{1}{\nu} \right)^{\frac{\varepsilon}{\varepsilon-1}} Y_t^{\frac{1}{\varepsilon}} \left(\frac{1}{\nu} \right)^{-\frac{1}{\varepsilon-1}} Y_t(h)^{-\frac{1}{\varepsilon}} \\ &= \left(\frac{1}{\nu} \right) Y_t^{\frac{1}{\varepsilon}} Y_t(h)^{-\frac{1}{\varepsilon}} \end{aligned}$$

, which is available for *h*’ as follows:

$$\lambda P_t(h') = \left(\frac{1}{\nu} \right) Y_t^{\frac{1}{\varepsilon}} Y_t(h')^{-\frac{1}{\varepsilon}}.$$

Combining both of them yields:

$$\frac{P_t(h)}{P_t(h')} = \left[\frac{Y_t(h)}{Y_t(h')} \right]^{-\frac{1}{\varepsilon}}$$

, which can be rewritten as:

$$Y_t(h) = \left[\frac{P_t(h)}{P_t(h')} \right]^{-\varepsilon} Y_t(h').$$

Plugging the previous expression into the definition of $C_{H,t}$ yields:

$$Y_t = \left\{ \left(\frac{1}{\nu}\right) \int_0^{\nu} \left[\left(\frac{P_t(h)}{P_t(h')}\right)^{-\varepsilon} Y_t(h') \right]^{\frac{\varepsilon-1}{\varepsilon}} dh \right\}^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \left(\frac{1}{\nu}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} P_t(h')^{\varepsilon-1} Y_t(h')^{\frac{\varepsilon-1}{\varepsilon}} dh \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

$$= \left(\frac{1}{\nu}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} dh P_t(h')^{\varepsilon-1} Y_t(h')^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \left(\frac{1}{\nu}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left[\int_0^{\nu} P_t(h)^{-(\varepsilon-1)} dh \right]^{\frac{\varepsilon}{\varepsilon-1}} P_t(h')^{\varepsilon} Y_t(h')$$

Notice that:

$$P_{H,t}^{-\varepsilon} = \left(\frac{1}{\nu}\right)^{-\frac{\varepsilon}{1-\varepsilon}} \left[\int_0^{\nu} P_t(h)^{1-\varepsilon} dh \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

Plugging this expression into the previous expression yields:

$$Y_t = \left(\frac{1}{\nu}\right)^{\frac{\varepsilon}{\varepsilon-1}} P_{H,t}^{-\varepsilon} \left(\frac{1}{\nu}\right)^{-\frac{\varepsilon}{\varepsilon-1}} P_t(h')^{\varepsilon} Y_t(h').$$

$$= P_{H,t}^{-\varepsilon} P_t(h')^{\varepsilon} Y_t(h')$$

The previous expression can be applicable any good *h*. Then, we have:

$$Y_t(h) = \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} Y_t. \text{ (A-1-6)}$$

Combining both of them with the definitions of the PPI indices yields:

$$Y_t(h) = \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} Y_{H,t}, \text{ (A-1-41)}$$

$$Y_t(f) = \left(\frac{P_t(f)}{P_{F,t}}\right)^{-\varepsilon} Y_t^*. \text{ (A-1-42)}$$

Now we consider firms' maximization problem following Gali (2015). The firms' maximization problem is given by:

$$\max_{\tilde{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \left\{ \Lambda_{t,t+k} \left(\frac{1}{P_{t+k}}\right) \left[\tilde{P}_{H,t} Y_{t+k|t} - \mathbb{C}_{t+k}\left(Y_{t+k|t}\right) \right] \right\},$$

with $Y_{t+k|t} \equiv \left(\frac{\tilde{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} Y_{t+k}$ and $\Lambda_{t,t+k} \equiv \mathrm{Q}_{t,t+k}\left(\frac{P_{t+k}}{P_t}\right) = \beta^k\left(\frac{\lambda_{t+k}}{\lambda_t}\right)$ being the real discount factor where $\mathrm{Q}_{t,t+k}$ denotes the price of a one period discount bond paying off one unit of currency *H* ($\mathrm{Q}_{t,t+1} = \frac{1}{1+i_t}$). The previous expression can be rewritten as :

$$\max_{\tilde{P}_{H,t}} \left\{ \begin{array}{l} \Lambda_{t,t}\left(\frac{1}{P_t}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t}}\right)^{-\varepsilon} Y_t - \mathbb{C}_t\left(\left(\frac{\tilde{P}_{H,t}}{P_{H,t}}\right)^{-\varepsilon} Y_t\right)\right] \\ +\theta\Lambda_{t,t+1}\left(\frac{1}{P_{t+1}}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+1}}\right)^{-\varepsilon} Y_{t+1} - \mathbb{C}_{t+1}\left(\left(\frac{\tilde{P}_{H,t}}{P_{H,t+1}}\right)^{-\varepsilon} Y_{t+1}\right)\right] \\ +\theta^2\Lambda_{t,t+2}\left(\frac{1}{P_{t+2}}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+2}}\right)^{-\varepsilon} Y_{t+2} - \mathbb{C}_{t+2}\left(\left(\frac{\tilde{P}_{H,t}}{P_{H,t+2}}\right)^{-\varepsilon} Y_{t+2}\right)\right] + \cdots \end{array} \right\}$$

The FONC for firms is given by:

$$\begin{array}{l} \Lambda_{t,t}\left(\frac{1}{P_t}\right)\left[(1-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon}P_{H,t}^{\varepsilon}Y_t - MC_{t|t}^{n}(-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon-1}P_{H,t}^{\varepsilon}Y_t\right] \\ +\theta\Lambda_{t,t+1}\left(\frac{1}{P_{t+1}}\right)\left[(1-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon}P_{H,t+1}^{\varepsilon}Y_{t+1} - MC_{t+1|t}^{n}(-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon-1}P_{H,t+1}^{\varepsilon}Y_{t+1}\right] \\ +\theta^2\Lambda_{t,t+2}\left(\frac{1}{P_{t+2}}\right)\left[(1-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon}P_{H,t+2}^{\varepsilon}Y_{t+2} - MC_{t+2|t}^{n}(-\varepsilon)\tilde{P}_{H,t}^{-\varepsilon-1}P_{H,t+2}^{\varepsilon}Y_{t+2}\right] + \cdots = 0 \end{array},$$

which can be rewritten as:

$$\begin{array}{l} \Lambda_{t,t}\left(\frac{1}{P_t}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t}}\right)^{-\varepsilon} Y_t - \frac{\varepsilon}{\varepsilon-1}MC_{t|t}^{n}\left(\frac{\tilde{P}_{H,t}}{P_{H,t}}\right)^{-\varepsilon} Y_t\right] \\ +\theta\Lambda_{t,t+1}\left(\frac{1}{P_{t+1}}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+1}}\right)^{-\varepsilon} Y_{t+1} - \frac{\varepsilon}{\varepsilon-1}MC_{t+1|t}^{n}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+1}}\right)^{-\varepsilon} Y_{t+1}\right] \\ +\theta^2\Lambda_{t,t+2}\left(\frac{1}{P_{t+2}}\right)\left[\tilde{P}_{H,t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+2}}\right)^{-\varepsilon} Y_{t+2} - \frac{\varepsilon}{\varepsilon-1}MC_{t+2|t}^{n}\left(\frac{\tilde{P}_{H,t}}{P_{H,t+2}}\right)^{-\varepsilon} Y_{t+2}\right] + \cdots = 0 \end{array},$$

with $MC_{H,t+k|t}^{n} \equiv \mathbb{C}'_{t+k}\left(Y_{t+k|t}\right)$ being the nominal marginal cost in country *H* in units of currency *H*.

By using the definition $Y_{t+k|t} \equiv \left(\frac{\tilde{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} Y_{t+k}$, the previous expression can be rewritten as:

$$\begin{aligned}
&\Lambda_{t,t}\left(\frac{1}{P_t}\right)\left[\tilde{P}_{H,t}Y_{t|t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t|t}Y_{t|t}\right] \\
&+\theta\Lambda_{t,t+1}\left(\frac{1}{P_{t+1}}\right)\left[\tilde{P}_{H,t}Y_{t+1|t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+1|t}Y_{t+1|t}\right] \qquad , \\
&+\theta^2\Lambda_{t,t+2}\left(\frac{1}{P_{t+2}}\right)\left[\tilde{P}_{H,t}Y_{t+2|t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+2|t}Y_{t+2|t}\right]+\cdots=0
\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}
&\Lambda_{t,t}\left(\frac{1}{P_t}\right)Y_{t|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t|t}\right) \\
&+\theta\Lambda_{t,t+1}\left(\frac{1}{P_{t+1}}\right)Y_{t+1|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+1|t}\right) \qquad . \\
&+\theta^2\Lambda_{t,t+2}\left(\frac{1}{P_{t+2}}\right)Y_{t+2|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+2|t}\right)+\cdots=0
\end{aligned}$$

The previous expression can be compact expression as:

$$\sum_{k=0}^{\infty}\theta^k \mathrm{E}_t\left[\Lambda_{t,t+k}\left(\frac{1}{P_{t+k}}\right)Y_{t+k|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k|t}\right)\right]=0 \text{. (A-1-43)}$$

Similarly, the following is applied in country *F*:

$$\sum_{k=0}^{\infty}\theta^k \mathrm{E}_t\left[\Lambda_{t,t+k}\left(\frac{1}{P^*_{t+k}}\right)Y^*_{t+k|t}\left(\tilde{P}^*_{F,t}-\frac{\varepsilon}{\varepsilon-1}MC^{n,*}_{t+k|t}\right)\right]=0 .$$

Nominal marginal cost is given by:

$$MC^n_{t+k|t}=\frac{W_{t+k}}{MPN_{t+k|t}},$$

with $MPN_{t+k} \equiv \frac{\partial Y_{t+k}}{\partial N_{t+k}}$ and $MPN_{t+k|t} \equiv \frac{\partial Y_{t+k}}{\partial N_{t+k|t}}$.

Plugging the definition of the real discount factor $\Lambda_{t,t+k} \equiv \mathrm{Q}_{t,t+k}\left(\frac{P_{t+k}}{P_t}\right)=\beta^k\left(\frac{\lambda_{t+k}}{\lambda_t}\right)$ into Eq.(A-1-43) yields:

$$\sum_{k=0}^{\infty}(\theta\beta)^k \mathrm{E}_t\left[\left(\frac{\lambda_{t+k}}{\lambda_t}\right)\left(\frac{1}{P_{t+k}}\right)Y_{t+k|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k|t}\right)\right]=0\,.$$

By multiplying $U_{C,t}$ both sides of the previous expression yields:

$$\sum_{k=0}^{\infty}(\theta\beta)^k \mathrm{E}_t\left[\left(\frac{\lambda_{t+k}}{P_{t+k}}\right)Y_{t+k|t}\left(\tilde{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k|t}\right)\right]=0\,,$$

which can be rewritten as:

$$\sum_{k=0}^{\infty}(\theta\beta)^k \mathrm{E}_t\left[\left(\frac{\lambda_{t+k}}{P_{t+k}}\right)Y_{t+k|t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\frac{MC^n_{t+k|t}}{P_{t+k}}\frac{P_{H,t+k}}{P_{H,t-1}}\right)\right]=0\,.$$

Let define $MC_{t+k|t}\equiv\frac{MC^n_{t+k|t}}{P_{H,t+k}}$ and $\Pi_{H,t-1,t+k}\equiv\frac{P_{H,t+k}}{P_{H,t-1}}$. Then, the previous expression can be rewritten as:

$$\sum_{k=0}^{\infty}(\theta\beta)^k \left[\left(\frac{\lambda_{t+k}}{P_{t+k}}\right)Y_{t+k|t}\left(\frac{\tilde{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}\Pi_{H,t-1,t+k}\right)\right]=0\,.$$

Let define $\tilde{X}_{H,t}\equiv\frac{\tilde{P}_{H,t}}{P_{H,t-1}}$. Then the previous expression can be rewritten as:

$$\sum_{k=0}^{\infty}(\theta\beta)^k \left\{\lambda_{t+k}Y_{t+k|t}\frac{P_{H,t-1}}{P_{t+k}}\left(\tilde{X}_{H,t}-\frac{\varepsilon}{\varepsilon-1}\Pi_{H,t-1,t+k}MC_{t+k|t}\right)\right\}=0\,.$$

Note that $P_{H,t-1}$ is multiplied on both sides of the previous expression. The previous expression can be rewritten as:
which can be rewritten as:

$$\begin{aligned}
&\lambda_t Y_{t|t}\frac{P_{H,t}}{P_t}\frac{P_{H,t-1}}{P_{H,t}}\left(\tilde{X}_{H,t}-\frac{\varepsilon}{\varepsilon-1}\frac{P_{H,t}}{P_{H,t-1}}MC_{t|t}\right)\\
&+\theta\beta\lambda_{t+1}Y_{t+1|t}\frac{P_{H,t+1}}{P_{t+1}}\frac{P_{H,t}}{P_{H,t+1}}\frac{P_{H,t-1}}{P_{H,t}}\left(\tilde{X}_{H,t}-\frac{\varepsilon}{\varepsilon-1}\frac{P_{H,t+1}}{P_{H,t}}\frac{P_{H,t}}{P_{H,t-1}}MC_{t+1|t}\right)\\
&+(\theta\beta)^2\lambda_{t+2}Y_{t+2|t}\frac{P_{H,t+2}}{P_{t+2}}\frac{P_{H,t+1}}{P_{H,t+2}}\frac{P_{H,t}}{P_{H,t+1}}\frac{P_{H,t-1}}{P_{H,t}}\left(\tilde{X}_{H,t}-\frac{\varepsilon}{\varepsilon-1}\frac{P_{H,t+2}}{P_{H,t+1}}\frac{P_{H,t+1}}{P_{H,t}}\frac{P_{H,t}}{P_{H,t-1}}MC_{t+2|t}\right)\\
&+\cdots=0
\end{aligned},$$

which can be rewritten as:

$$
\begin{aligned}
&\lambda_t Y_{t|t} g_H\left(S_t\right)^{-1} \Pi_{H,t}^{-1}\left(\tilde{X}_{H,t} - \frac{\varepsilon}{\varepsilon-1}\Pi_{H,t} MC_{t|t}\right) \\
&+\theta\beta\lambda_{t+1} Y_{t+1|t} g_H\left(S_{t+1}\right)^{-1} \Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\left(\tilde{X}_{H,t} - \frac{\varepsilon}{\varepsilon-1}\Pi_{H,t+1}\Pi_{H,t} MC_{t+1|t}\right) \\
&+\left(\theta\beta\right)^2 \lambda_{t+2} Y_{t+2|t} g_H\left(S_{t+2}\right)^{-1} \Pi_{H,t+2}^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\left(\begin{array}{l}\tilde{X}_{H,t} \\ -\frac{\varepsilon}{\varepsilon-1}\Pi_{H,t+2}\Pi_{H,t+1}\Pi_{H,t} MC_{t+2|t}\end{array}\right) \\
&+\cdots = 0
\end{aligned},
$$

with $g_H\left(S_t\right) \equiv \frac{P_t}{P_{H,t}}$ and $\frac{\partial g_H\left(S_t\right)}{\partial S_t} > 0$.

Similarly, we have:

$$
\begin{aligned}
&\lambda_t^* Y_{t|t}^* g_F^*\left(S_t\right)^{-1}\left(\Pi_{F,t}^*\right)^{-1}\left(\tilde{X}_{F,t}^* - \frac{\varepsilon}{\varepsilon-1}\Pi_{F,t}^* MC_{t|t}^*\right) \\
&+\theta\beta\lambda_{t+1}^* Y_{t+1|t}^* g_F^*\left(S_{t+1}\right)^{-1}\left(\Pi_{F,t+1}^*\right)^{-1}\left(\Pi_{F,t}^*\right)^{-1}\left(\tilde{X}_{F,t}^* - \frac{\varepsilon}{\varepsilon-1}\Pi_{F,t+1}^*\Pi_{F,t}^* MC_{t+1|t}^*\right) \\
&+\left(\theta\beta\right)^2 \lambda_{t+2}^* Y_{t+2|t}^* g_F^*\left(S_{t+2}\right)^{-1}\left(\Pi_{F,t+2}^*\right)^{-1}\left(\Pi_{F,t+1}^*\right)^{-1}\left(\Pi_{F,t}^*\right)^{-1}\left(\begin{array}{l}\tilde{X}_{F,t}^* \\ -\frac{\varepsilon}{\varepsilon-1}\Pi_{F,t+2}^*\Pi_{F,t+1}^*\Pi_{F,t}^* MC_{t+2|t}^*\end{array}\right) \\
&+\cdots = 0
\end{aligned}
$$

with $\tilde{X}_{F,t}^* \equiv \frac{\tilde{P}_{F,t}^*}{P_{F,t-1}^*}$, $g_F^*\left(S_t\right) \equiv \frac{P_t^*}{P_{F,t}^*}$ and $\frac{\partial g_F^*\left(S_t\right)}{\partial S_t} < 0$.

The previous expression can be rewritten as:

$$
\begin{aligned}
&\lambda_t Y_{H,t|t} S_t^{-\nu}\Pi_{H,t}^{-1}\tilde{X}_{H,t} - \frac{\varepsilon}{\varepsilon-1}\lambda_t Y_{t|t} S_t^{-\nu} MC_{t|t} \\
&+\theta\beta\lambda_{t+1} Y_{t+1|t} S_{t+1}^{-\nu}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\tilde{X}_{H,t} \\
&-\frac{\varepsilon}{\varepsilon-1}\theta\beta\lambda_{t+1} Y_{t+1|t} S_{t+1}^{-\nu} MC_{t+1|t} \\
&+\left(\theta\beta\right)^2 \lambda_{t+2} Y_{t+2|t} S_{t+2}^{-\nu}\Pi_{H,t+2}^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\tilde{X}_{H,t} \\
&-\frac{\varepsilon}{\varepsilon-1}\left(\theta\beta\right)^2 \lambda_{t+2} Y_{t+2|t} S_{t+2}^{-\nu} MC_{t+2|t} \\
&+\cdots = 0
\end{aligned}.
$$

By moving the terms related to the marginal cost to the RHS yields:

$$\lambda_t Y_{H,t|t} g_H\left(S_t\right)^{-1}\Pi_{H,t}^{-1}\tilde{X}_{H,t}$$
$$+\theta\beta\lambda_{t+1} Y_{H,t+1|t} g_H\left(S_{t+1}\right)^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\tilde{X}_{H,t}$$
$$+(\theta\beta)^2\lambda_{t+2} Y_{H,t+2|t} g_H\left(S_{t+2}\right)^{-1}\Pi_{H,t+2}^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\tilde{X}_{H,t}$$
$$+\cdots$$
$$\frac{\varepsilon}{\varepsilon-1}\lambda_t Y_{H,t|t} g_H\left(S_t\right)^{-1} MC_{H,t|t}$$
$$= \quad +\frac{\varepsilon}{\varepsilon-1}\theta\beta\lambda_{t+1} Y_{H,t+1|t} g_H\left(S_{t+1}\right)^{-1} MC_{H,t+1|t}$$
$$+\frac{\varepsilon}{\varepsilon-1}(\theta\beta)^2\lambda_{t+2} Y_{H,t+2|t} g_H\left(S_{t+2}\right)^{-1} MC_{H,t+2|t}$$
$$+\cdots$$

which can be simplified as follows:

$$\tilde{X}_{H,t}\left\{\begin{array}{l}\lambda_t Y_{t|t} g_H\left(S_t\right)^{-1}\Pi_{H,t}^{-1}\\ +\theta\beta\lambda_{t+1} Y_{t+1|t} g_H\left(S_{t+1}\right)^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}\\ +(\theta\beta)^2\lambda_{t+2} Y_{t+2|t} g_H\left(S_{t+2}\right)^{-1}\Pi_{H,t+2}^{-1}\Pi_{H,t+1}^{-1}\Pi_{H,t}^{-1}+\cdots\end{array}\right\}.$$
$$=\frac{\varepsilon}{\varepsilon-1}\left\{\begin{array}{l}\lambda_t Y_{t|t} g_H\left(S_t\right)^{-1} MC_{t|t}\\ +\theta\beta\lambda_{t+1} Y_{t+1|t} g_H\left(S_{t+1}\right)^{-1} MC_{t+1|t}\\ +(\theta\beta)^2\lambda_{t+2} Y_{t+2|t} g_H\left(S_{t+2}\right)^{-1} MC_{t+2|t}+\cdots\end{array}\right\}$$

Then, we have:

$$\tilde{X}_{H,t}\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k} Y_{t+k|t} g_H\left(S_{t+k}\right)^{-1}\prod_{h=0}^{k}\Pi_{H,t+h}^{-1} \quad ,$$
$$=\frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k} Y_{t+k|t} g_H\left(S_{t+k}\right)^{-1} MC_{t+k|t}$$

or:

$$\tilde{X}_{H,t}=\frac{\frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k} Y_{t+k|t} g_H\left(S_{t+k}\right)^{-1} MC_{t+k|t}}{\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k} Y_{t+k|t} g_H\left(S_{t+k}\right)^{-1}\prod_{h=0}^{k}\Pi_{H,t+h}^{-1}}. \quad \text{(A-1-44)}$$

Similarly, we have:

$$\tilde{X}_{F,t}^{*}=\frac{\frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k}^{*} Y_{t+k|t}^{*} g_F^{*}\left(S_{t+k}\right)^{-1} MC_{t+k|t}^{*}}{\sum_{k=0}^{\infty}(\theta\beta)^k\lambda_{t+k}^{*} Y_{t+k|t}^{*} g_F^{*}\left(S_{t+k}\right)^{-1}\prod_{h=0}^{k}\Pi_{F,t+h}^{-1}}. \quad \text{(A-1-45)}$$

Plugging

$$
\begin{aligned}
MC^{n}_{t+k|t} &= \frac{W_{t+k}}{MPN_{t+k|t}} \\
&= W_{t+k}\left(\frac{\partial Y_{t+k}}{\partial N_{t+k|t}}\right)^{-1} \\
&= W_{t+k}\left[(1-\alpha)A_{t+k}N^{-\alpha}_{t+k|t}\right]^{-1} \\
&= \frac{W_{t+k}}{(1-\alpha)A_{t+k}}N^{\alpha}_{t+k|t}
\end{aligned},
$$

into the definition of the marginal cost yields:

$$MC_{t+k} \equiv \frac{W_{t+k}}{P_{H,t+k}(1-\alpha)A_{t+k}}N^{\alpha}_{t+k}. \text{ (A-1-46)}$$

Similarly, we have:

$$MC^{*}_{t+k} \equiv \frac{W^{*}_{t+k}}{P^{*}_{F,t+k}(1-\alpha)A_{t+k}}\left(N^{*}_{t+k}\right)^{\alpha}. \text{ (A-1-47)}$$

1.5 Market Clearing Condition

The market clearing condition is given by:

$$Y_t(j) = C_t(j) + G_t(j),$$

where $G_t(j)$ denotes the government purchase for the good produced by firm *j*, respectively.

Let define $G_t \equiv \left[\left(\frac{1}{\nu}\right)\int_0^{\nu} G_t(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $G^{*}_t \equiv \left[\left(\frac{1}{1-\nu}\right)\int_{\nu}^{1} G^{*}_t(f)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$.

The, by solving the maximization problems, we have:

$$G_t(h) = \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} G_t, \text{ (A-1-48)}$$

$$G^{*}_t(f) = \left(\frac{P^{*}_t(f)}{P^{*}_{F,t}}\right)^{-\varepsilon} G^{*}_t. \text{ (A-1-49)}$$

Plugging Eqs. (A-1-6), (A-1-13), (A-1-41) and (A-1-48) into the market clearing condition yields:

$$\left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} Y_t = \frac{1}{\nu}\left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} \nu\left(\frac{P_{H,t}}{P_t}\right)^{-\zeta} C_t + \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} G_t$$
$$= \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} \left(\frac{P_{H,t}}{P_t}\right)^{-\zeta} C_t + \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} G_t,$$

which can be rewritten as:

$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\zeta} C_t^W + G_t$. (A-1-50)

Plugging Eqs. (A-1-7), (A-1-14), (A-1-41) and (A-1-49) into the market clearing condition yields:

$$\left(\frac{P_t(f)}{P_{F,t}}\right)^{-\varepsilon} Y_t^* = \frac{1}{1-\nu}\left(\frac{P_t(f)}{P_{F,t}}\right)^{-\varepsilon} (1-\nu)\left(\frac{P_{F,t}}{P_t}\right)^{-\zeta} C_t^W + \left(\frac{P_t^*(f)}{P_{F,t}^*}\right)^{-\varepsilon} G_t^*,$$

which can be rewritten as:

$Y_t^* = \left(\frac{P_{F,t}}{P_t}\right)^{-\zeta} C_t^W + G_t^*$. (A-1-51)

1.6 Government Budget Constraint

The government budget constraints is given by:

$P_{H,t}G_t + B_{t-1}(1+i_{t-1}) = P_t TR_{H,t} + B_t + \Delta M_t$, (A-1-52)

$P_{F,t}^* G_t^* + B_{t-1}^*(1+i_{t-1}^*) = P_t^* TR_t^* + B_t^* + \Delta M_t^*$, (A-1-53)

Dividing both side of Eqs.(A-1-52) and (A-1-53) by P_t and P_t^* yields:

$$\frac{P_{H,t}}{P_t} G_t + \mathrm{B}_{t-1}(1+i_{t-1})\frac{P_{t-1}}{P_t} = TR_t + \mathrm{B}_t + \frac{\Delta M_t}{P_t},$$

$$\frac{P_{F,t}^*}{P_t^*} G_t^* + \mathrm{B}_{t-1}^*(1+i_{t-1}^*)\frac{P_{t-1}^*}{P_t^*} = TR_t^* + \mathrm{B}_t^* + \frac{\Delta M_t^*}{P_t^*}.$$

Let define the (ex-post) real (consumption) interest rate $R_t \equiv (1+i_t)\frac{P_t}{P_{t+1}}$ and $R_t^* \equiv (1+i_t^*)\frac{P_t^*}{P_{t+1}^*}$. Then the previous expression can be rewritten as:

$$\frac{P_{H,t}}{P_t}G_t + \mathrm{B}_{t-1}R_{t-1} = TR_t + \mathrm{B}_t + \frac{\Delta M_t}{P_t}, \text{ (A-1-54)}$$

$$\frac{P_{F,t}^*}{P_t^*}G_t^* + \mathrm{B}_{t-1}^*R_{t-1}^* = TR_t^* + \mathrm{B}_t^* + \frac{\Delta M_t^*}{P_t^*}. \text{ (A-1-55)}$$

The level of seignorage, expressed as a fraction of steady state output can be approximated as:

$$\begin{aligned}\frac{\Delta M_t}{P_t}\frac{1}{Y} &= \frac{\Delta M_t}{P_t}\frac{M_{t-1}}{M_{t-1}}\frac{P_{t-1}}{P_{t-1}}\frac{1}{Y} \\ &= \frac{\Delta M_t}{M_{t-1}}\frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t}\frac{1}{Y}, \text{ (A-1-56)} \\ &= \frac{\Delta M_t}{M_{t-1}}\frac{P_{t-1}}{P_t}L_{t-1}\frac{1}{Y}\end{aligned}$$

$$\frac{\Delta M_t^*}{P_t^*}\frac{1}{Y} = \frac{\Delta M_t^*}{M_{t-1}^*}\frac{P_{t-1}^*}{P_t^*}L_{t-1}^*\frac{1}{Y}, \text{ (A-1-57)}$$

Quantity theory of money implies as follows:

$MV = PY$,

which can be rewritten as:

$V^{-1} = \frac{L}{Y}$.

Plugging the previous expression into Eqs.(A-1-56) yields:

$$\frac{\Delta M_t}{P_t}\frac{1}{Y} = \chi\Delta m_t, \text{ (A-1-58)}$$

$$\frac{\Delta M_t^*}{P_t^*}\frac{1}{Y} = \chi\Delta m_t^*, \text{ (A-1-59)}$$

with $\chi \equiv V^{-1}$ being the inverse of income velocity of money. Note that Eq.(A-1-58) ignore changes in the inflation and the deviation of the real money balance from its steady state.

If we do not ignore them, we have:

$$\begin{aligned}\frac{\Delta M_t}{P_t}\frac{1}{Y} &= \frac{\Delta M_t}{M_{t-1}}\frac{P_{t-1}}{P_t}L_{t-1}\frac{1}{Y} \\ &= \ln\left(\frac{M_t}{M_{t-1}}\right)\Pi_{t-1}^{-1}\frac{L_{t-1}}{L}\frac{L}{Y} \\ &= \chi\ln\left(\frac{M_t}{M_{t-1}}\right)\Pi_{t-1}^{-1}\frac{L_{t-1}}{L}\end{aligned}$$

1.7 Trade Balance

Similar to Gali and Monaceli (2005, RES), We define the real trade balance as follows:

$$\frac{NX_t}{P_{H,t}} = Y_t - g(S_t)C_t - G_t \text{, (A-1-60)}$$

$$\frac{NX_t}{P^*_{F,t}} = Y^*_t - g^*(S_t)C^*_t - G^*_t \text{ (A-1-61)}$$

$$NX_t = -E_t NX^*_t .$$

with $g(S_t) \equiv \frac{P_t}{P_{H,t}}$ and $g^*_t(S_t) \equiv \frac{P_t}{P_{F,t}}$.

2 The Steady State

We focus on equilibria where the state variables follow paths that are close to a deterministic stationary equilibrium, in which $\Pi_t = 1$, $\Delta M_{H,t} = \Delta M_{F,t} = 0$ and $TR_{H,t} = TR_{F,t} = TR$. Further, we assume $A_{H,t} = A_{F,t} = 1$ and $G_{H,t} = G_{F,t} = 0$.

Eqs.(A-1-30) and (A-1-14) implies as follows:

$$\beta = \frac{1}{1+i} = \frac{1}{1+i^*} \text{, (A-2-1)} = \frac{1}{1+\rho}$$

which implies $E_t = E$.

The PPP implies:

$Q = 1$.

Eqs.(A-1-32) and (A-1-33) implies that:

$$\frac{W}{P} = \frac{V_n}{U_c} \text{, (A-2-2)}$$

$$\frac{W^*}{P^*} = \frac{V^*_n}{U^*_c} \text{. (A-2-3)}$$

Eqs.(A-1-34) and (A-1-35) implies as follows:

$$\frac{U_l}{U_c} = \beta\rho \text{, (A-2-4)}$$

$$\frac{U_l^*}{U_c^*} = \beta\rho \text{. (A-2-5)}$$

Eqs.(2-4) and (2-5) imply $U_l = U_l^*$ so that:

$L = L^*$.

Eqs.(A-1-44) and (A-1-45) implies:

$$1 = \frac{\frac{\varepsilon}{\varepsilon-1}\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^{-1}\right)^{-1}\right]Yg(S)^{-1}MC}{\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^{-1}\right)^{-1}\right]Yg(S)^{-1}},$$

$$= \frac{\varepsilon}{\varepsilon-1}MC$$

$$1 = \frac{\frac{\varepsilon}{\varepsilon-1}\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^{-1}\right)^{-1}\right]Y^*g^*(S)^{-1}MC^*}{\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^{-1}\right)^{-1}\right]Y^*g^*(S)^{-1}},$$

$$= \frac{\varepsilon}{\varepsilon-1}MC^*$$

which implies:

$$MC = MC^* = \frac{\varepsilon-1}{\varepsilon} = \mathrm{M}^{-1}\text{. (A-2-6)}$$

Eqs.(A-1-46) and (A-1-47) implies:

$$\begin{aligned}\mathrm{M}^{-1} &= \frac{W}{P_H(1-\alpha)}N^\alpha \\ &= \frac{W^*}{P_F^*(1-\alpha)}\left(N^*\right)^\alpha\end{aligned}.$$

Plugging Eqs.(A-2-2) and (A-2-3) into the previous expression yields:

$$\begin{aligned}\mathrm{M}^{-1} &= \frac{V_n}{U_c}\frac{P}{P_H}\frac{N^\alpha}{1-\alpha} \\ &= \frac{V_n^*}{U_c^*}\frac{P^*}{P_F^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\end{aligned}\text{. (A-2-7)}$$

Eqs.(A-1-50) and (A-1-51) imply:

$$\frac{P}{P_H} = \left(\frac{Y}{C}\right)^{\frac{1}{\zeta}}\text{, (A-2-8)}$$

$$\frac{P}{P_F}=\left(\frac{Y^*}{C^*}\right)^{\frac{1}{\zeta}}. \text{ (A-2-9)}$$

Plugging Eqs.(A-2-8) and (A-2-9) into Eq.(A-2-7) yields:

$$\begin{aligned}\mathrm{M}^{-1}&=\frac{V_n}{U_c}\left(\frac{Y}{C}\right)^{\frac{1}{\zeta}}\frac{N^{\alpha}}{1-\alpha}\\&=\frac{V_n^*}{U_c^*}\left(\frac{Y^*}{C^*}\right)^{\frac{1}{\zeta}}\frac{\left(N^*\right)^{\alpha}}{1-\alpha}\end{aligned}. \text{ (A-2-10)}$$

Eqs.(A-1-39) and (A-1-40) yields:

$Y=N^{1-\alpha}$, (A-2-11)

$Y^*=\left(N^*\right)^{1-\alpha}$. (A-2-12)

Plugging Eqs.(A-2-11) and (A-2-12) yields:

$$\begin{aligned}\mathrm{M}^{-1}&=\frac{V_nN^{\frac{1+\alpha(\zeta-1)}{\zeta}}}{U_cC^{\frac{1}{\zeta}}}\frac{1}{1-\alpha}\\&=\frac{V_n^*\left(N^*\right)^{\frac{1+\alpha(\zeta-1)}{\zeta}}}{U_c^*\left(C^*\right)^{\frac{1}{\zeta}}}\frac{1}{1-\alpha}\end{aligned},$$

which implies $V_{N_H}N_H^{\frac{1+\alpha(\zeta-1)}{\zeta}}=V_{N_F}N_F^{\frac{1+\alpha(\zeta-1)}{\zeta}}$ because of $C=C^*$ ($\lambda=\lambda^*$). That is:

$N=N^*$.

Plugging the previous expression into Eq.(A-2-11) and (A-2-12) yields:

$Y=Y^*$. (A-2-13)

Then, Eqs.(A-1-2-11) and (A-1-2-12) boils down to:

$Y=N^{1-\alpha}$.

Plugging Eq.(A-2-13) into Eqs.(A-2-8) and (A-2-9) yields:

$$\left(\frac{Y}{C}\right)^{\frac{1}{\zeta}}=\frac{P}{P_H}=\frac{P}{P_F}, \text{ (A-2-14)}$$

which implies $P_H=P_F$. That is:

$S=1$.

Eq.(A-1-5) implies:

$$
\begin{aligned}
P &= \left[\nu P_H^{1-\varsigma} + (1-\nu) P_F^{1-\varsigma}\right]^{\frac{1}{1-\varsigma}} \\
&= \left\{P_H^{1-\varsigma}\left[\nu + (1-\nu)\right]\right\}^{\frac{1}{1-\varsigma}} \quad , \\
&= P_H
\end{aligned}
$$

which implies:

$P = P_H = P_F$.

Plugging the previous expression into Eq.(A-2-14) yields:

$Y = C$.

Eqs.(A-1-54) and (A-1-55) implies:

$\mathrm{B}_H \rho = TR$,

$\mathrm{B}_F \rho = TR$,

which imply:

$\mathrm{B}_H = \mathrm{B}_F = \mathrm{B}$.

As mentioned, Eqs.(A-1-46) and (A-1-47) implies:

$$
\begin{aligned}
\mathrm{M}^{-1} &= \frac{W}{P_H (1-\alpha)} N^{\alpha} \\
&= \frac{W^*}{P_F E^{-1} (1-\alpha)} N^{\alpha}
\end{aligned} \text{. (A-2-15)}
$$

where we use $N = N^*$. Eq.(A-2-15) implies:

$W = EW^*$,

because of $P_H = P_F = P$.

Finally, Eq.(A-2-10) implies:

$$\frac{V_n}{U_c} = \mathrm{M}^{-1} \frac{1-\alpha}{N^{\alpha}} \text{. (A-2-16)}$$

3 Log-linearization of the Model

3.1 Log-linearizing Some Entities

Taking logarithm Eq.(A-1-5) yields:

$p_t = \nu p_{H,t} + (1-\nu) p_{F,t}$, (A-3-1)

with $p_t \equiv \log P_t$, $p_{H,t} \equiv \log P_{H,t}$ and $p_{F,t} \equiv \log P_{F,t}$.

Subtracting one-period delayed equality of Eq.(A-3-1) from Eq.(A-3-1) itself yields:

$\pi_t = \nu \pi_{H,t} + (1-\nu) \pi_{F,t}$, (A-3-2): π_t

with:

$\pi_t \equiv \log \Pi_t$,
$\pi_{H,t} = \hat{p}_{H,t} - \hat{p}_{H,t-1}$, (A-3-3): $p_{H,t}$
$\pi_{F,t} \equiv \hat{p}_{F,t} - \hat{p}_{F,t-1}$, (A-3-4): $\pi_{F,t}$
Taking logarithm Eq.(A-1-37) yields:

$$p_t = e_t + p_t^*,$$

with $p_t^* \equiv \log P_t^*$ and $e_t = \log E_t$.

Subtracting one-period delayed equality of the previous expression from the previous expression itself yields:

$\pi_t = e_t - e_{t-1} + \pi_t^*$, (A-3-5): π_t^*

with $\pi_t^* \equiv p_t^* - p_{t-1}^*$.

The LOOP can be log-linearized as follows:

$p_{F,t} = e_t + p_{F,t}^*$. (A-3-6): $\hat{p}_{F,t}$

with $p_{F,t}^* \equiv \log P_{F,t}^*$.

Subtracting one-period delayed equality of Eq.(A-3-6) from Eq.(A-3-6) itself yields:

$$\pi_{F,t} = e_t - e_{t-1} + \pi_{F,t}^*.$$

with $\pi_{F,t}^* = \hat{p}_{F,t}^* - \hat{p}_{F,t-1}^*$, (A-3-7): $\hat{p}_{F,t}^*$

Taking logarithm the definition of the TOT yields:
$s_t = p_{F,t} - p_{H,t}$. (A-3-8): s_t
Log-linearizing the UIP yields:

$\hat{i}_t = \hat{i}_t^* + e_{t+1} - e_t$, (A-3-9): e_t

with $\hat{i}_t = \log\left(\frac{1+i_t}{1+\rho}\right)$ and $\hat{i}_t^* = \log\left(\frac{1+i_t^*}{1+\rho}\right)$.

3.2 Log-linearizing the Market Clearing Condition

Subtracting $p_{H,t}$ from both sides of Eq.(A-3-1) yields:

$$p_t - p_{H,t} = -(1-\nu)p_{H,t} + (1-\nu)p_{F,t} \\ = (1-\nu)s_t$$. (A-3-10)

Subtracting $p_{F,t}$ from both sides of Eq.(A-3-1) yields:

$$p_t - p_{F,t} = \nu p_{H,t} - \nu p_{F,t} \\ = -\nu s_t$$. (A-3-11)

Total derivative of Eq.(A-1-50) is given by:

$$dY_t = (-\zeta)\left[\frac{dP_{H,t}}{P} + P_H(-1)P^{-2}dP_t\right] + dC_t^W + dG_t \\ = -\zeta\frac{dP_{H,t}}{P} + \zeta\frac{dP_t}{P} + dC_t^W + dG_{H,t}$$.

By dividing both sides of previous expression by $Y_H = C$, we have:

$$\hat{y}_t = -\zeta\hat{p}_{H,t} + \zeta\hat{p}_t + \hat{c}_t^W + \hat{g}_{H,t},$$

with $\hat{y}_{H,t} \equiv \log\left(\frac{Y_{H,t}}{Y_H}\right)$, $\hat{c}_t^W \equiv \log\left(\frac{C_t^W}{C^W}\right)$ and $\hat{g}_t \equiv \log\left(\frac{G_t}{Y}\right)$.

Plugging Eq.(A-3-10) into the previous expression yields:

$$y_t = \zeta(1-\nu)s_t + \hat{c}_t^W + \hat{g}_t, \; (\hat{y}_t)$$

which is (logarithmic) market clearing in country *H*.

Total derivative of Eq.(A-1-51) is given by:

$$dY_t^* = (-\zeta)\left[\frac{1}{P}dP_{F,t} + P_F(-1)P^{-2}dP_t\right] + dC_t^W + dG_t^* \\ = -\zeta\frac{dP_{F,t}}{P_F} + \zeta\frac{dP_t}{P} + dC_t^W + dG_t^*$$.

By dividing both sides of previous expression by $Y = C$, we have:

$$\hat{y}_t^* = -\zeta\hat{p}_{F,t} + \zeta\hat{p}_t + \hat{c}_t^W + \hat{g}_t^*,$$

with $\hat{y}_t^* \equiv \log\left(\frac{Y_t^*}{Y^*}\right)$ and $\hat{g}_t^* \equiv \log\left(\frac{G_t^*}{Y^*}\right)$ where

Plugging Eq.(A-3-11) into the previous expression yields:

$$\hat{y}_t^* = -\zeta\nu s_t + \hat{c}_t^W + \hat{g}_t^*, \; (\hat{y}_{F,t})$$

which is (logarithmic) market clearing in country *F*.

Subtracting (logarithmic) market clearing in country *F* from (logarithmic) market clearing in country *H* yields:

$$s_t = \frac{1}{\zeta}\left(\hat{y}_t - \hat{y}_t^*\right) - \frac{1}{\zeta}\left(\hat{g}_t - \hat{g}_t^*\right). \text{ (A-3-12)}$$

Combining (logarithmic) market clearing in country *F* from (logarithmic) market clearing in country *H* yields:

$$\nu \hat{y}_t + (1-\nu)\hat{y}_t^* = \nu \hat{c}_t + (1-\nu)\hat{c}_t^* + \nu \hat{g}_t + (1-\nu)\hat{g}_t^*. \text{ (A-3-13)}$$

3.3 Log-linearizing Euler Equation

Total derivative of Eq.(A-1-30) is given by:

$$dU_{C,t} = U_C \beta d\left(1 + r_{H,t}\right) + U_C(-1)d\Pi_{t+1} + dU_{c,t+1} + U_C dZ_{t+1} - U_C dZ_t.$$

$$U_{C,t} = \beta\left(1 + r_{H,t}\right)\Pi_{t+1}^{-1} U_{C,t+1}\left(\frac{Z_{t+1}}{Z_t}\right)$$

Note that $\beta = (1+\rho)^{-1}$. Thus:

$$dU_{C,t} = U_C \frac{d\left(1 + r_{H,t}\right)}{1+\rho} + U_C(-1)d\Pi_{t+1} + dU_{C,t+1} + U_C dZ_{t+1} - U_C dZ_t.$$

Dividing both sides of the previous expression by U_c yields:

$$\frac{dU_{C,t}}{U_C} = \frac{d\left(1 + r_{H,t}\right)}{1+\rho} - d\Pi_{t+1} + \frac{dU_{C,t+1}}{U_C} + dZ_{t+1} - dZ_t,$$

which can be rewritten as:

$$\log\left(\frac{U_{C,t}}{U_C}\right) = \log\left(\frac{1 + r_{H,t}}{1+\rho}\right) - \log\Pi_{t+1} + \log\left(\frac{U_{C,t+1}}{U_C}\right) + \log Z_{t+1} - \log Z_t.$$

Let define $\hat{\xi}_t \equiv \log\left(\frac{U_{C,t}}{U_C}\right)$ and $\hat{\rho}_t \equiv -\log\left(\frac{Z_{t+1}}{Z_t}\right)$. Then, the previous expression can be rewritten as:

$$\hat{\xi}_t = \hat{\xi}_{t+1} + \hat{r}_{H,t} - \pi_{t+1} - \hat{\rho}_t, \text{ (A-3-15)}$$

which is a class of log-linearized Euler equation.

Total derivative of Eq.(A-1-31) is given by:

$$dU_{c,t}^{*}=U_{c}^{*}\beta\frac{d\left(1+i_{t}^{*}\right)}{1+\rho}-U_{c}^{*}d\Pi_{t+1}^{*}+dU_{c,t+1}^{*}+U_{c}^{*}dZ_{t+1}^{*}-U_{c}^{*}dZ_{t}^{*}.$$

$$U_{c,t}^{*}=\beta\left(1+i_{t}^{*}\right)\left(\Pi_{t+1}^{*}\right)^{-1}U_{c,t+1}^{*}\left(\frac{Z_{t+1}^{*}}{Z_{t}^{*}}\right). \text{ (A-1-31)}$$

Dividing both sides of the previous expression by U_c yields:

$$\hat{\xi}_{t}^{*}=\hat{\xi}_{t+1}^{*}+\hat{i}_{t}^{*}-\pi_{t+1}^{*}-\hat{\rho}_{t}^{*}. \text{ (A-3-16)}$$

3.4 Log-linearizing Marginal Utility of Consumption

Marginal utility of consumption can be rewritten as:

$$U_{c,t}=U_{l,t}\left(\frac{U_{l,t}}{U_{c,t}}\right)^{-1}. \text{ (A-3-17)}$$

Total derivative of Eq.(3-12) is given by:

$$\begin{aligned}dU_{c,t}&=\left\{U_{ll}\left(\frac{U_{L_l}}{U_{C}}\right)^{-1}+U_{l}(-1)\left(\frac{U_{l}}{U_{c}}\right)^{-2}\left[\frac{1}{U_{c}}U_{ll}+U_{l}(-1)U_{c}^{-2}\frac{\partial U_{c}}{\partial L}\right]\right\}dL_{t}\\&+\left\{U_{lc}\left(\frac{U_{l}}{U_{c}}\right)^{-1}+U_{l}(-1)\left(\frac{U_{Ll}}{U_{c}}\right)^{-2}\left[\frac{1}{U_{c}}U_{lc}+U_{l}(-1)U_{c}^{-2}\frac{\partial U_{c}}{\partial C}\right]\right\}dC_{t}\\&=U_{l}^{2}\left(\frac{U_{c}}{U_{l}}\right)^{2}\frac{1}{U_{c}^{2}}U_{cl}L\frac{dL_{t}}{L}+\left[U_{lc}\frac{U_{c}}{U_{l}}C-U_{l}\left(\frac{U_{c}}{U_{l}}\right)^{2}\left(\frac{U_{lc}}{U_{c}}C-\frac{U_{l}}{U_{c}}\frac{U_{cc}}{U_{c}}C\right)\right]\frac{dC_{t}}{C}\\&=U_{cl}L\frac{dL_{t}}{L}+\left[\frac{U_{lc}}{U_{c}}U_{c}C-U_{c}\left(\frac{U_{lc}}{U_{l}}C-\frac{U_{cc}}{U_{c}}C\right)\right]\frac{dC_{t}}{C}\end{aligned}.$$

Dividing both sides of the previous expression by U_C yields:

$$\begin{aligned}\frac{dU_{c,t}}{U_{C}}&=\frac{U_{cl}}{U_{c}}L\frac{dL_{t}}{L}+\left(\frac{U_{lc}}{U_{l}}C-\frac{U_{lc}}{U_{l}}C+\frac{U_{cc}}{U_{c}}C\right)\frac{dC_{t}}{C}\\&=\frac{U_{cl}}{U_{c}}L_{i}\frac{dL_{t}}{L}+\frac{U_{cc}}{U_{c}}C\frac{dC_{t}}{C}\end{aligned},$$

$$\hat{\xi}_{t}=\upsilon\hat{l}_{t}-\sigma\hat{c}_{t}, \text{ (A-3-18)}$$

Similarly:

$\hat{\xi}_t^* = \upsilon \hat{l}_t^* - \sigma \hat{c}_t^*$. (A-3-19)

3.5 Deriving the FONC for Domestic Producers

Total derivative of Eq.(A-1-44) yields:

$$d\tilde{X}_{H,t} = \frac{\varepsilon}{\varepsilon-1}\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]^{-1}\left[\begin{matrix} dMC_{t|t}+\theta\beta dMC_{t+1|t} \\ +(\theta\beta)^2 dMC_{t+2|t}+\cdots \end{matrix}\right] + \frac{\varepsilon}{\varepsilon-1}(MC)\left[d\Pi_{H,t}+\theta\beta d\Pi_{H,t+1}+(\theta\beta)^2 d\Pi_{H,t+2}+\cdots\right]$$

Note that $1+\theta\beta+(\theta\beta)^2+\cdots = \frac{1}{1-\theta\beta}$ and $MC = \frac{\varepsilon-1}{\varepsilon}$. Then, the previous expression can be rewritten as:

$$d\tilde{X}_{H,t} = (1-\theta\beta)\left[\frac{dMC_{t|t}}{MC}+\theta\beta\frac{dMC_{t+1|t}}{MC}+(\theta\beta)^2\frac{dMC_{t+2|t}}{MC}+\cdots\right], + \left[d\Pi_{H,t}+\theta\beta d\mathrm{E}_t\left(\Pi_{H,t+1}\right)+(\theta\beta)^2 d\mathrm{E}_t\left(\Pi_{H,t+2}\right)+\cdots\right]$$

which can be rewritten as:

$$\tilde{p}_{H,t}-p_{H,t-1} = (1-\theta\beta)\left[\widehat{mc}_{t|t}+\theta\beta\widehat{mc}_{t+1|t}+(\theta\beta)^2\widehat{mc}_{t+2|t}+\cdots\right], + \left[\pi_{H,t}+\theta\beta\pi_{H,t+1}+(\theta\beta)^2\pi_{H,t+2}+\cdots\right]$$

with $\widehat{mc}_{t+k|t} \equiv \log\left(\frac{MC_{t+k|t}}{MC}\right) = mc_{t+k|t} - mc$, $mc_t \equiv \log MC_t$ and $mc \equiv \log MC = -\log\left(\frac{\varepsilon-1}{\varepsilon}\right)$.

Previous expression can be rewritten as:

$$\begin{aligned} \tilde{p}_{H,t}-p_{H,t-1} &= (1-\theta\beta)\left[\widehat{mc}_{t|t}+\theta\beta\widehat{mc}_{t+1|t}+(\theta\beta)^2\widehat{mc}_{t+2|t}+\cdots\right] \\ &\quad+\left[p_{H,t}-p_{H,t-1}+\theta\beta\left(p_{H,t+1}-p_{H,t}\right)+(\theta\beta)^2\left(p_{H,t+2}-p_{H,t+1}\right)+\cdots\right] \\ &= (1-\theta\beta)\left[\widehat{mc}_{t|t}+\theta\beta\widehat{mc}_{t+1|t}+(\theta\beta)^2\widehat{mc}_{t+2|t}+\cdots\right] \\ &\quad+\left[(1-\theta\beta)p_{H,t}+\theta\beta(1-\theta\beta)p_{H,t+1}+(\theta\beta)^2(1-\theta\beta)p_{H,t+2}+\cdots\right]-p_{H,t-1} \\ &= (1-\theta\beta)\left(\widehat{mc}_{t|t}+p_{H,t}\right)+\theta\beta(1-\theta\beta)\left(\widehat{mc}_{t+1|t}+p_{H,t+1}\right) \\ &\quad+(\theta\beta)^2(1-\theta\beta)\left(\widehat{mc}_{t+2|t}+p_{H,t+2}\right)+\cdots-p_{H,t-1} \end{aligned} \quad . \text{(A-3-20)}$$

Note that:

$$\widehat{mc}_{t+k|t}+p_{H,t+k}=mc_{t+k|t}+p_{H,t+k}-\log MC$$
$$=mc^{n}_{t+k|t}-\log MC$$
$$=mc^{n}_{t+k|t}-\log\left(\frac{\varepsilon}{\varepsilon-1}\right)^{-1} \quad . \text{(A-3-21)}$$
$$=mc^{n}_{t+k|t}+\log\left(\frac{\varepsilon}{\varepsilon-1}\right)$$
$$=mc^{n}_{t+k|t}+\mu$$

Plugging Eq.(A-3-21) into Eq.(A-3-20) yields:

$$\begin{aligned}\tilde{p}_{H,t}-p_{H,t-1}&=(1-\theta\beta)\left(mc^{n}_{t|t}+\mu\right)+\theta\beta(1-\theta\beta)\left(mc^{n}_{t+1|t}+\mu\right)\\&\quad+(\theta\beta)^{2}(1-\theta\beta)\left(mc^{n}_{t+2|t}+\mu\right)+\cdots-p_{H,t-1}\\&=(1-\theta\beta)\left[mc^{n}_{t|t}+\theta\beta mc^{n}_{t+1|t}+(\theta\beta)^{2}mc^{n}_{t+2|t}+\cdots\right]-p_{H,t-1}\\&\quad+(1-\theta\beta)\left[1+\theta\beta+(\theta\beta)^{2}+\cdots\right]\mu\\&=\mu+(1-\theta\beta)\left[mc^{n}_{t|t}+\theta\beta mc^{n}_{t+1|t}+(\theta\beta)^{2}mc^{n}_{t+2|t}+\cdots\right]-p_{H,t-1}\end{aligned} \quad ,$$

which can be rewritten as:

$$\tilde{p}_{H,t}=\mu+(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^{k}mc^{n}_{t+k|t} \quad . \text{(A-3-22)}$$

(Corresponding to Eq.11 in Chap. 3, Gali, 2015)

Eq.(A-3-20) can be rewritten as:

$$\begin{aligned}\tilde{p}_{H,t}-p_{H,t-1}&=(1-\theta\beta)\left[\widehat{mc}_{t|t}+\theta\beta\widehat{mc}_{t+1|t}+(\theta\beta)^{2}\widehat{mc}_{t+2|t}+\cdots\right]\\&\quad+\left[p_{H,t}-p_{H,t-1}+\theta\beta\left(p_{H,t+1}-p_{H,t}\right)+(\theta\beta)^{2}\left(p_{H,t+2}-p_{H,t+1}\right)+\cdots\right]\\&=(1-\theta\beta)\left[\widehat{mc}_{t|t}+\theta\beta\widehat{mc}_{t+1|t}+(\theta\beta)^{2}\widehat{mc}_{t+2|t}+\cdots\right]\\&\quad+\left[(1-\theta\beta)p_{H,t}+\theta\beta(1-\theta\beta)p_{H,t+1}+(\theta\beta)^{2}(1-\theta\beta)p_{H,t+2}+\cdots\right]-p_{H,t-1}\\&=(1-\theta\beta)\left(\widehat{mc}_{t|t}+p_{H,t}\right)+\theta\beta(1-\theta\beta)\left(\widehat{mc}_{t+1|t}+p_{H,t+1}\right)\\&\quad+(\theta\beta)^{2}(1-\theta\beta)\left(\widehat{mc}_{t+2|t}+p_{H,t+2}\right)+\cdots-p_{H,t-1}\end{aligned} \quad .$$

Eq.(A-3-20) can be rewritten as:

$$\tilde{p}_{H,t}-p_{H,t-1}=(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^{k}\widehat{mc}_{t+k|t}+\sum_{k=0}^{\infty}(\theta\beta)^{k}\pi_{H,t+k} \quad . \text{(A-3-23)}$$

Eq.(A-3-23) can be rewritten as:

$$\tilde{p}_{H,t}-p_{H,t-1}=(1-\theta\beta)\widehat{mc}_{t|t}+\pi_{H,t}+(1-\theta\beta)\sum_{k=1}^{\infty}(\theta\beta)^{k}\widehat{mc}_{t+k|t}+\sum_{k=1}^{\infty}(\theta\beta)^{k}\pi_{H,t+k} \quad . \text{(A-3-24)}$$

Forwarding Eq.(A-3-24) one period yields:

$$\tilde{p}_{H,t+1} - p_{H,t} = \frac{1-\theta\beta}{\theta\beta}\sum_{k=1}^{\infty}(\theta\beta)^k \widehat{mc}_{t+k|t} + \frac{1}{\theta\beta}\sum_{k=1}^{\infty}(\theta\beta)^k \pi_{H,t+k} .$$

Multiplying $\theta\beta$ on the both sides of the previous expression yields:

$$\theta\beta\left(\tilde{p}_{H,t+1} - p_{H,t}\right) = (1-\theta\beta)\sum_{k=1}^{\infty}(\theta\beta)^k \widehat{mc}_{t+k|t} + \sum_{k=1}^{\infty}(\theta\beta)^k \pi_{H,t+k} .$$

Plugging the previous expression into Eq.(A-3-24) yields:

$$\tilde{p}_{H,t} - p_{H,t-1} = \theta\beta\left(\tilde{p}_{H,t+1} - p_{H,t}\right) + (1-\theta\beta)\widehat{mc}_{t|t} + \pi_{H,t} . \text{ (A-3-25)}$$

Calvo-pricing's transitory equation is given by:

$$P_{H,t} = \left[\theta P_{H,t-1}^{1-\varepsilon} + (1-\theta)\tilde{P}_{H,t}^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

Log-linearizing the previous expression around the steady state yields:

$$p_{H,t} = \theta p_{H,t-1} + (1-\theta)\tilde{p}_{H,t} .$$

Subtracting $p_{H,t-1}$ from the both sides of the previous expression yields:

$$\pi_{H,t} = (1-\theta)\left(\tilde{p}_{H,t} - p_{H,t-1}\right) . \text{ (A-3-26)}$$

Plugging Eq.(A-3-26) into Eq.(A-3-25) yields:

$$\frac{1}{1-\theta}\pi_{H,t} = \theta\beta\frac{1}{1-\theta}\pi_{H,t+1} + (1-\theta\beta)\widehat{mc}_{H,t|t} + \pi_{H,t} ,$$

which can be rewritten as:

$$\frac{\theta}{1-\theta}\pi_{H,t} = \theta\beta\frac{1}{1-\theta}\pi_{H,t+1} + (1-\theta\beta)\widehat{mc}_{H,t|t} .$$

Multiplying both sides of the previous expression by $\frac{1-\theta}{\theta}$ yields:

$$\pi_{H,t} = \beta\pi_{H,t+1} + \frac{(1-\theta\beta)(1-\theta)}{\theta}\widehat{mc}_{t|t} .$$

Let assume $Y_{t+k|t} = A_{t+k}N_{t+k|t}^{1-\alpha}$. Then, the (nominal) marginal cost for an individual firm that last set its price is given by:

$$
\begin{aligned}
MC^n_{t+k|t} &= W_{t+k}\frac{\partial N_{t+k|t}}{\partial Y_{t+k|t}} \\
&= W_{t+k} MPN^{-1}_{t+k|t} \\
&= W_{t+k}\left(\frac{\partial Y_{t+k|t}}{\partial N_{t+k|t}}\right)^{-1} \\
&= W_{t+k}\left(\frac{\partial A_{t+k} N^{1-\alpha}_{t+k|t}}{\partial N_{t+k|t}}\right)^{-1} \\
&= W_{t+k}\left[(1-\alpha) A_{t+k} N^{-\alpha}_{t+k|t}\right]^{-1}
\end{aligned}
.
$$

Note that the (nominal) average marginal cost is given by:

$$
\begin{aligned}
MC^n_{H,t+k} &= W_{H,t+k}\frac{\partial N_{H,t+k}}{\partial Y_{H,t+k}} \\
&= W_{H,t+k}\left(\frac{\partial Y_{H,t+k}}{\partial N_{H,t+k}}\right)^{-1} \\
&= W_{H,t+k}\left[(1-\alpha) A_{H,t+k} N^{-\alpha}_{H,t+k}\right]^{-1}
\end{aligned}
$$

Total derivative of the (nominal) marginal cost for an individual firm that last set its price is given by:

$$
\begin{aligned}
dMC^n_{t+k|t} &= \frac{N^\alpha}{1-\alpha} dW_{t+k} + W(-1)\left[(1-\alpha)N^{-\alpha}\right]^{-2}(1-\alpha)(-\alpha)N^{-\alpha-1} dN_{t+k|t} \\
&= \frac{WN^\alpha}{1-\alpha}\frac{dW_{t+k}}{W} + W\frac{1}{(1-\alpha)N^{-\alpha}}\frac{(1-\alpha)\alpha N^{-\alpha}}{(1-\alpha)N^{-\alpha}}\frac{dN_{t+k|t}}{N} \\
&= \frac{WN^\alpha}{1-\alpha}\frac{dW_{t+k}}{W} + \frac{WN^\alpha}{1-\alpha}\alpha\frac{dN_{t+k|t}}{N}
\end{aligned}
.
$$

Dividing both sides of the previous expression by MC^n yields:

$$
\begin{aligned}
\frac{dMC^n_{t+k|t}}{MC^n} &= \frac{1-\alpha}{WN^\alpha}\left(\frac{WN^\alpha}{1-\alpha}\frac{dW_{t+k}}{W} + \frac{WN^\alpha}{1-\alpha}\alpha\frac{dN_{t+k|t}}{N}\right) \\
&= \frac{dW_{t+k}}{W} + \alpha\frac{dN_{t+k|t}}{N}
\end{aligned}
,
$$

which can be rewritten as:

$mc^n_{t+k|t} = w_{t+k} + \alpha \hat{n}_{t+k|t}$. (A-3-27)

Log-linearization of the average (log) marginal cost is given by:

$mc^n_{t+k} = w_{t+k} + \alpha \hat{n}_{t+k}$. (A-3-28)

Subtracting Eq.(A-3-28) from Eq.(A-3-27) yields:

$mc^n_{t+k|t} - mc^n_{t+k} = \alpha\left(\hat{n}_{t+k|t} - \hat{n}_{t+k}\right)$,

which can be rewritten as:

$mc^n_{t+k|t} = mc^n_{t+k} + \alpha\left(\hat{n}_{t+k|t} - \hat{n}_{t+k}\right)$.

Plugging the logarithmic production function $\hat{y}_{t+k|t} = a_{t+k} + (1-\alpha)\hat{n}_{t+k|t}$ and $\hat{y}_{t+k} = a_{t+k} + (1-\alpha)\hat{n}_{t+k}$ into the previous expression yields:

$mc^n_{t+k|t} = mc^n_{t+k} + \dfrac{\alpha}{1-\alpha}\left(\hat{y}_{t+k|t} - \hat{y}_{t+k}\right)$.

Plugging logarithmic demand function of $Y_{t+k|t} \equiv \left(\dfrac{\tilde{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} Y_{t+k}$ which is given by $\hat{y}_{t+k|t} = -\varepsilon\left(\tilde{p}_{H,t} - p_{H,t+k}\right) + \hat{y}_{t+k}$ into the previous expression yields:

$mc^n_{t+k|t} = mc^n_{t+k} - \dfrac{\alpha\varepsilon}{1-\alpha}\left(\tilde{p}_{H,t} - p_{H,t+k}\right)$. (A-3-29)

Plugging Eq.(A-3-29) into Eq.(A-3-22) yields:

$$\begin{aligned}\tilde{p}_{H,t} &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left[mc^n_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(\tilde{p}_{H,t} - p_{H,t+k}\right)\right],\\ &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(mc^n_{t+k} + \frac{\alpha\varepsilon}{1-\alpha}p_{H,t+k}\right) - \frac{\alpha\varepsilon}{1-\alpha}\tilde{p}_{H,t}\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}\frac{(1-\alpha)+\alpha\varepsilon}{1-\alpha}\tilde{p}_{H,t} &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(mc^n_{t+k} + \frac{\alpha\varepsilon}{1-\alpha}p_{H,t+k}\right)\\ &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(p_{H,t+k} - \mu_{t+k} + \frac{\alpha\varepsilon}{1-\alpha}p_{H,t+k}\right),\\ &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(-\mu_{t+k} + \frac{(1-\alpha)+\alpha\varepsilon}{1-\alpha}p_{H,t+k}\right)\\ &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(\mu - \mu_{t+k} + \frac{(1-\alpha)+\alpha\varepsilon}{1-\alpha}p_{H,t+k}\right)\end{aligned}$$

where we use the definition of the (log) desired markup $\mu_t \equiv -\left(mc^n_t - p_{H,t}\right)$ which is (log) inverse of the real marginal cost.

Let define $\hat{\mu}_t \equiv \mu_t - \mu$ being the deviation between the average and desired marginal cost. Plugging the definition into the previous expression yields:

$$\tilde{p}_{H,t}=(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left[p_{H,t+k}-\frac{1-\alpha}{(1-\alpha)+\alpha\varepsilon}\hat{\mu}_{t+k}\right]. \quad \text{(A-3-30)}$$

with $\Theta\equiv\frac{1-\alpha}{(1-\alpha)+\alpha\varepsilon}$.

Eq.(A-3-30) can be rewritten as:

$$\begin{aligned}
\tilde{p}_{H,t}-p_{H,t-1} &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\left(p_{H,t+k}-\Theta\hat{\mu}_{t+k}\right)-p_{H,t-1} \\
&= (1-\theta\beta)\left[p_{H,t}+\theta\beta p_{H,t+1}+(\theta\beta)^2 p_{H,t+2}+\cdots\right]-p_{H,t-1} \\
&\quad -(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k} \\
&= -p_{H,t-1}+\left[p_{H,t}+\theta\beta p_{H,t+1}+(\theta\beta)^2 p_{H,t+2}+\cdots\right] \\
&\quad -\theta\beta\left[p_{H,t}+\theta\beta p_{H,t+1}+(\theta\beta)^2 p_{H,t+2}+\cdots\right] \\
&\quad -(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k} \\
&= p_{H,t}-p_{H,t-1}+\theta\beta\left(p_{H,t+1}-p_{H,t}\right)+(\theta\beta)^2\left(p_{H,t+2}-p_{H,t+1}\right) \\
&\quad +(\theta\beta)^3\left(p_{H,t+3}-p_{H,t+2}\right)+\cdots \\
&\quad -(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k} \qquad\qquad , \text{(A-3-31)} \\
&= \pi_{H,t}+\theta\beta\pi_{H,t+1}+(\theta\beta)^2\pi_{H,t+2}+(\theta\beta)^3\pi_{H,t+3}+\cdots \\
&\quad -(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k} \\
&= \sum_{k=0}^{\infty}(\theta\beta)^k\pi_{H,t+k}-(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k}
\end{aligned}$$

Forwarding Eq.(A-3-31) one period yields:

$$\begin{aligned}
\tilde{p}_{H,t+1}-p_{H,t} &= \pi_{H,t+1}+\theta\beta\pi_{H,t+2}+(\theta\beta)^2\pi_{H,t+3}+(\theta\beta)^3\pi_{H,t+4}+\cdots \\
&\quad -(1-\theta\beta)\Theta\left[\hat{\mu}_{t+1}+\theta\beta\hat{\mu}_{t+2}+(\theta\beta)^2\hat{\mu}_{t+3}+(\theta\beta)^3\hat{\mu}_{t+4}+\cdots\right] \\
&= \frac{1}{\theta\beta}\sum_{k=1}^{\infty}(\theta\beta)^k\pi_{H,t+k}-\frac{(1-\theta\beta)\Theta}{\theta\beta}\sum_{k=1}^{\infty}(\theta\beta)^k\hat{\mu}_{t+k}
\end{aligned}$$

Multiplying $\theta\beta$ on the both sides of the previous expression yields:

$$\theta\beta\left(\tilde{p}_{H,t+1}-p_{H,t}\right)=\sum_{k=1}^{\infty}(\theta\beta)^k\pi_{H,t+k}-(1-\theta\beta)\sum_{k=1}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k}.$$

Plugging the previous expression into Eq.(A-3-31) yields:

$$\begin{aligned}\tilde{p}_{H,t}-p_{H,t-1}&=\pi_{H,t}+\theta\beta\pi_{H,t+1}+(\theta\beta)^2\pi_{H,t+2}+(\theta\beta)^3\pi_{H,t+3}+\cdots\\&\quad-(1-\theta\beta)\left[\Theta\hat{\mu}_t+\theta\beta\Theta\hat{\mu}_{t+1}+(\theta\beta)^2\hat{\mu}_{t+2}+(\theta\beta)^3\hat{\mu}_{t+3}\right]\\&=\pi_{H,t}-(1-\theta\beta)\Theta\hat{\mu}_t+\sum_{k=1}^{\infty}(\theta\beta)^k\pi_{H,t+k}-(1-\theta\beta)\sum_{k=1}^{\infty}(\theta\beta)^k\Theta\hat{\mu}_{t+k}\\&=\pi_{H,t}-(1-\theta\beta)\Theta\hat{\mu}_t+\theta\beta\left(\tilde{p}_{H,t+1}-p_{H,t}\right)\end{aligned}.$$

Plugging Eq.(A-3-26) into the previous expression yields:

$$\frac{1}{1-\theta}\pi_{H,t}=\pi_{H,t}-(1-\theta\beta)\Theta\hat{\mu}_t+\theta\beta\frac{1}{1-\theta}\pi_{H,t+1},$$

which can be rewritten as:

$$\pi_{H,t}=\frac{1-\theta}{\theta}\left[-(1-\theta\beta)\Theta\hat{\mu}_t+\theta\beta\frac{1}{1-\theta}\pi_{H,t+1}\right].$$

Then, we have:

$\pi_{H,t}=\beta\pi_{H,t+1}-\kappa\hat{\mu}_t$, (A-3-32) ($\pi_{H,t}$)

with $\kappa\equiv\frac{(1-\theta\beta)(1-\theta)}{\theta}\Theta$.

Similar to Eq.(A-3-32), we have:

$\pi^*_{F,t}=\beta\pi^*_{F,t+1}-\kappa\hat{\mu}^*_t$, (A-3-33) ($\pi^*_{F,t}$)

with $\hat{\mu}^*_t\equiv\mu^*_t-\mu$ and . $\mu_{F,t}\equiv-\left(mc^{*n}_{F,t}-p^*_{F,t}\right)$.

3.6 Log-linearization of Intra-temporal Optimality Condition

Dividing both sides of Eq.(A-1-32) by $P_t/P_{H,t}$ yields:

$$\frac{W_t}{P_{H,t}}=\frac{V_{n,t}}{U_{c,t}}\frac{P_t}{P_{H,t}}.$$

Plugging the previous expression into Eq.(A-1-46) yields:

$$MC_t=\frac{V_{n,t}}{U_{c,t}}\frac{P_t}{P_{H,t}}\frac{N_t^{\alpha}}{1-\alpha}.$$

Total derivative of the previous expression is given by:

$$dMC_t = \left(\frac{1}{U_c}\frac{N^\alpha}{1-\alpha}\frac{\partial V_n}{\partial N} + \frac{V_n}{U_c}\frac{\alpha}{1-\alpha}\frac{N^\alpha}{N}\right)dN_t - U_c^{-2}V_n\frac{N^\alpha}{1-\alpha}dU_{c,t} + \frac{V_n}{U_c}\frac{N^\alpha}{1-\alpha}\left(\frac{dP_t}{P} - \frac{dP_{H,t}}{P}\right)$$
$$= \frac{V_n}{U_c}\frac{N^\alpha}{1-\alpha}\left(\frac{V_{nn}N}{V_n} + \alpha\right)\frac{dN_t}{N} - \frac{V_n}{U_c}\frac{N^\alpha}{1-\alpha}\frac{dU_{c,t}}{U_c} + \frac{V_n}{U_c}\frac{N^\alpha}{1-\alpha}\left(\frac{dP_t}{P} - \frac{dP_{H,t}}{P}\right).$$

Plugging Eq.(A-2-16) into the previous expression yields:

$$dMC_t = \frac{1-\alpha}{N^\alpha \mathrm{M}}\frac{N^\alpha}{1-\alpha}\left(\frac{V_{nn}N}{V_n} + \alpha\right)\frac{dN_t}{N} - \frac{1-\alpha}{N^\alpha \mathrm{M}}\frac{N^\alpha}{1-\alpha}\frac{dU_{c,t}}{U_c} + \frac{1-\alpha}{N^\alpha \mathrm{M}}\frac{N^\alpha}{1-\alpha}\left(\frac{dP_t}{P} - \frac{dP_{H,t}}{P}\right)$$
$$= \mathrm{M}^{-1}\left(\frac{V_{nn}N}{V_n} + \alpha\right)\frac{dN_t}{N} - \mathrm{M}^{-1}\frac{dU_{c,t}}{U_c} + \mathrm{M}^{-1}\left(\frac{dP_t}{P} - \frac{dP_{H,t}}{P}\right).$$

Multiplying both sides of the previous expression by $\mathrm{M} = MC^{-1}$ yields:

$$\frac{dMC_t}{MC} = \left(\frac{V_{nn}N}{V_n} + \alpha\right)\frac{dN_t}{N} - \frac{dU_{c,t}}{U_c} + \left(\frac{dP_t}{P} - \frac{dP_{H,t}}{P}\right),$$

which can be rewritten as:

$$\log\left(\frac{MC_t}{MC}\right) = \left(\frac{V_{nn}N}{V_n} + \alpha\right)\log\left(\frac{N_t}{N}\right) - \log\left(\frac{U_{c,t}}{U_c}\right) + \left[\log\left(\frac{dP_t}{P}\right) - \log\left(\frac{dP_{H,t}}{P}\right)\right].$$

By using the definition of $\widehat{mc}_t \equiv \log\left(\frac{MC_t}{MC}\right)$, $\hat{n}_t \equiv \log\left(\frac{N_t}{N}\right)$, $\hat{\xi}_t \equiv \log\left(\frac{U_{c,t}}{U_c}\right)$ and $\varphi \equiv \frac{V_{nn}N}{V_n}$, the previous expression can be rewritten as:

$\widehat{mc}_t = (\varphi + \alpha)\hat{n}_t - \hat{\xi}_t + (1-\nu)s_t$, (A-3-34)

where we use Eq.(A-3-10).
Eq.(A-3-21) implies as follows:

$$\widehat{mc}_t + p_{H,t} = mc_t^n + \mu$$
$$= -\mu_t + p_{H,t} + \mu, \text{ or } \widehat{mc}_t = -\hat{\mu}_t,$$
$$= -\hat{\mu}_t + p_{H,t}$$

where we use $mc_t^n = -\mu_t + p_{H,t}$ which is derived by the definition of the desired markup. Plugging the previous expression into Eq.(A-3-34) yields:

$\hat{\mu}_t = \hat{\xi}_t - (\varphi + \alpha)\hat{n}_t - (1-\nu)s_t$.

Plugging the (log) production function $\hat{y}_{H,t} = (1-\alpha)\hat{n}_{H,t}$ derived from Eq.(1-21) into the previous expression yields:

$\hat{\mu}_t = \hat{\xi}_t - \frac{\varphi + \alpha}{1-\alpha}\hat{y}_t - (1-\nu)s_t$. (A-3-35) ($\hat{\mu}_{H,t}$)

Dividing both sides of Eq.(A-1-33) by $\frac{P_t^*}{P_{F,t}^*}$ yields:

$$\frac{W_t^*}{P_{F,t}^*} = \frac{V_{n,t}^*}{U_{c,t}^*}\frac{P_t^*}{P_{F,t}^*}.$$

Plugging the previous expression into Eq.(A-1-47) yields:

$$MC_{F,t}^* = \frac{V_{n,t}^*}{U_{c,t}^*}\frac{P_t^*}{P_{F,t}^*}\frac{\left(N_t^*\right)^\alpha}{1-\alpha}.$$

Total derivative of the previous expression is given by:

$$\begin{aligned} dMC_t^* &= \left[\frac{1}{U_c^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\frac{\partial V_n^*}{\partial N^*} + \frac{V_n^*}{U_c^*}\frac{\alpha}{1-\alpha}\frac{\left(N^*\right)^\alpha}{N^*}\right]dN_t^* - \left(U_c^*\right)^{-2}V_n^*\frac{\left(N^*\right)^\alpha}{1-\alpha}dU_{c,t}^* \\ &+ \frac{V_n^*}{U_c^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\left(\frac{dP_t^*}{P^*} - \frac{dP_{F,t}^*}{P^*}\right) \\ &= \frac{V_n^*}{U_c^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\left(\frac{V_{nn}^*N^*}{V_n^*} + \alpha\right)\frac{dN_t^*}{N^*} - \frac{V_n^*}{U_c^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\frac{dU_{c,t}}{U_c} + \frac{V_n^*}{U_c^*}\frac{\left(N^*\right)^\alpha}{1-\alpha}\left(\frac{dP_t^*}{P^*} - \frac{dP_{F,t}^*}{P^*}\right) \end{aligned}.$$

Plugging Eq.(A-2-16) into the previous expression yields:

$$dMC_t^* = \mathrm{M}^{-1}\left(\frac{V_{nn}^*N^*}{V_n^*} + \alpha\right)\frac{dN_t^*}{N^*} - \mathrm{M}^{-1}\frac{dU_{c,t}^*}{U_c^*} + \mathrm{M}^{-1}\left(\frac{dP_t^*}{P^*} - \frac{dP_{F,t}^*}{P^*}\right).$$

Multiplying both sides of the previous expression by $\mathrm{M} = MC^{-1}$ yields:

$$\log\left(\frac{MC_t^*}{MC^*}\right) = \left(\frac{V_{nn}^*N^*}{V_n^*} + \alpha\right)\log\left(\frac{N_t^*}{N^*}\right) - \log\left(\frac{U_{c,t}^*}{U_c^*}\right) + \left[\log\left(\frac{dP_t^*}{P^*}\right) - \log\left(\frac{dP_{F,t}^*}{P^*}\right)\right].$$

By using the definition of $\widehat{mc}_t^* \equiv \log\left(\frac{MC_t^*}{MC^*}\right)$, $\hat{n}_t^* \equiv \log\left(\frac{N_t^*}{N^*}\right)$ and $\varphi \equiv \frac{V_{nn}N}{V_n}$, the previous expression can be rewritten as:

$\widehat{mc}_t^* = (\varphi + \alpha)\hat{n}_t^* - \hat{\xi}_t^* - \nu s_t$, (A-3-36)

where we use Eq.(A-3-11).

Plugging $\hat{\mu}_t^* \equiv \mu_t^* - \mu$ and . $\mu_t^* \equiv -\left(mc_t^{*n} - p_{F,t}^*\right)$ into Eq.(A-3-36) yields:

$$\hat{\mu}_t^* = -(\varphi + \alpha)\hat{n}_t^* + \hat{\xi}_t^* + \nu s_t .$$

Plugging the (log) production function $\hat{y}_t^* = (1-\alpha)\hat{n}_t^*$ derived from Eq.(1-21) into the previous expression yields:

$\hat{\mu}_t^* = -\dfrac{\varphi + \alpha}{1-\alpha}\hat{y}_t^* + \hat{\xi}_t^* + \nu s_t$. (A-3-37) ($\hat{\mu}_{F,t}$)

3.7 Deriving the LM Equation

Eq.(A-1-34) can be rewritten as:

$$\frac{U_{l,t}}{U_{c,t}} = \frac{i_t}{1+i_t} .$$

Multiplying —1 on both sides of the previous expression yields:

$$-\frac{U_{l,t}}{U_{c,t}} = -\frac{i_t}{1+i_t}$$

Summing 1 both sides of the previous expression yields:

$$\begin{aligned} 1-\frac{U_{l,t}}{U_{c,t}} &= 1-\frac{i_t}{1+i_t} \\ &= \frac{1}{1+i_t} \end{aligned} .$$

Raise both sides of the previous expression to power −1 yields:

$$1+i_t = \left(1-\frac{U_{l,t}}{U_{c,t}}\right)^{-1} .$$

Total derivative of the previous expression yields:

$$d\left(1+i_t\right)=-\left(1-\frac{U_l}{U_c}\right)^{-2}\left[-U_l\left(-1\right)U_c^{-2}U_{cc}-\frac{1}{U_c}U_{cl}\right]dC_t+\left[-\frac{1}{U_c}U_{ll}-U_l\left(-1\right)U_c^{-2}U_{cl}\right]dL_t$$

$$=-\left(1+\rho\right)^{-2}\left[\left(\frac{U_l}{U_c}\frac{U_{cc}}{U_c}-\frac{U_{lc}}{U_c}\right)dC_t+\left(-\frac{U_l}{U_c}\frac{U_{ll}}{U_l}+\frac{U_l}{U_c}\frac{U_{lc}}{U_c}\right)dL_t\right]$$

$$=-\left(1+\rho\right)^{-2}\frac{U_l}{U_c}\left[\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\frac{dC_t}{C}+\left(-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L\right)\frac{dL_t}{L}\right]$$

$$=-\left(1+\rho\right)\rho\left[\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\frac{dC_t}{C}+\left(-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L\right)\frac{dL_t}{L}\right].$$

Dividing both sides of the previous expression by $1+\rho$ yields:

$$\frac{d\left(1+i_t\right)}{1+\rho}=-\rho\left[\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\frac{dC_t}{C}+\left(-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L\right)\frac{dL_t}{L}\right]$$

$$=-\rho\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\frac{dC_t}{C}-\rho\left(-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L\right)\frac{dL_t}{L},$$

which can be rewritten as:

$$\left(-\frac{U_{ll}}{U_l}L_i+\frac{U_{lc}}{U_c}L\right)\frac{dL_t}{L}=-\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\frac{dC_t}{C}-\frac{1}{\rho}\frac{d\left(1+i_t\right)}{1+\rho}. \quad \text{(A-3-38)}$$

Iff $-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L-\left[-\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right)\right]=0$, (A-3-39)

$$-\frac{U_{ll}}{U_l}L+\frac{U_{lc}}{U_c}L=-\left(\frac{U_{cc}}{U_c}C-\frac{U_c}{U_l}\frac{U_{lc}}{U_c}C\right), \quad \text{(A-3-40)}$$

is applicable.

Let assume $U\left(C,L\right)=\frac{1}{1-\varpi}\left(C^{1-\vartheta}L^{\vartheta}\right)^{1-\varpi}$. Then, we have:

$$U_c = (1-\vartheta) h' \left(\frac{L}{C}\right)^{\vartheta},$$

$$U_{cc} = (1-\vartheta)\left[-\sigma(1-\vartheta)-\vartheta\right] h' \left(\frac{L}{C}\right)^{\vartheta} C^{-1},$$

$$U_{cl} = \vartheta(1-\vartheta)(1-\varpi) h' \left(\frac{L}{C}\right)^{\vartheta} L^{-1}, \qquad \text{(A-3-41)}$$

$$U_l = \vartheta h' \left(\frac{L}{C}\right)^{\vartheta-1},$$

$$U_{ll} = \vartheta\left[-\sigma\vartheta-(1-\vartheta)\right] h' \left(\frac{L}{C}\right)^{\vartheta-1} L^{-1},$$

with $h' \equiv \left(C^{1-\vartheta} L_i^{\vartheta}\right)^{-\varpi}$.

Plugging Eq.(A-3-41) into the RHS of Eq.(A-3-40) yields:

$$-\left(\frac{U_{CC}}{U_C}C - \frac{U_C}{U_{L_i}}\frac{U_{L_iC}}{U_C}C\right) = -\left[-\sigma(1-\vartheta)-\vartheta-(1-\vartheta)(1-\varpi)\right] \\ =1 \qquad \text{(A-3-42)}$$

Plugging Eq.(A-3-41) into the LHS of Eq.(A-3-40) yields:

$$-\frac{U_{ll}}{U_l}L + \frac{U_{lc}}{U_c}L = -\left[-\varpi\vartheta-(1-\vartheta)\right] + \vartheta(1-\varpi) \\ =1 \qquad \text{(A-3-43)}$$

Plugging Eqs.(A-3-42) and (A-3-43) into the LHS of Eq.(A-3-39) yields:

1—1=0.

Thus, Eq.(A-3-40) is applicable. Plugging Eq.(A-3-40) into Eq.(A-3-38) yields:

$$\left(-\frac{U_{ll}}{U_l}L + \frac{U_{lc}}{U_c}L\right)\log\left(\frac{L_t}{L}\right) = -\left(-\frac{U_{ll}}{U_l}L + \frac{U_{lc}}{U_c}L\right)\log\left(\frac{C_t}{C}\right) - \frac{1}{\rho}\log\left(\frac{1+i_t}{1+\rho}\right),$$

where we use the fact that $\frac{dL_t}{L} = \log\left(\frac{L_t}{L}\right)$, $\log\left(\frac{C_t}{C}\right)$ and $\frac{d(1+i_t)}{1+\rho} = \log\left(\frac{1+i_t}{1+\rho}\right)$.

By using definitions $\sigma_l \equiv -\frac{U_{ll}}{U_l}L$ and $\upsilon \equiv \frac{U_{lc}}{U_c}L$, the previous expression can be rewritten as:

$$(\sigma_l + \upsilon)\hat{l}_t = (\sigma_l + \upsilon)\hat{c}_t - \frac{1}{\rho}\hat{i}_t,$$

which can be rewritten as:

$$\hat{l}_t = \hat{c}_t - \frac{1}{\rho(\sigma_l + \upsilon)}\hat{i}_t .$$

By using the definition $\varepsilon_{lc} \equiv \frac{1}{\sigma_l + \upsilon}$, the previous expression can be rewritten as:

$\hat{l}_t = \hat{c}_t - \eta\hat{i}_t$, (A-3-45) ($\hat{l}_t$)

with $\eta \equiv \frac{\varepsilon_{lc}}{\rho}$. Similarly,

$\hat{l}_t^* = \hat{c}_t^* - \eta\hat{i}_t^*$. (A-3-46) ($\hat{l}_t^*$)

3.8 Relationship between Changes in the Real Money Balance and Inflation

Taking logarithm the definition of the real money balance $L_t \equiv \frac{M_t}{P_t}$ yields:

$$\log L_t = \log M_t - \log P_t .$$

Subtracting one period delayed equality from the previous expression yields:

$$\log L_t - \log L_{t-1} = \log M_t - \log M_{t-1} - (\log P_t - \log P_{t-1}),$$

which can be rewritten as:

$$\log\left(\frac{L_t}{L}\right) - \log\left(\frac{L_{t-1}}{L}\right) = \log M_t - \log M_{t-1} - \left[\log\left(\frac{P_t}{P}\right) - \log\left(\frac{P_{t-1}}{P}\right)\right],$$

which can be rewritten as:

$\hat{l}_{t-1} = \hat{l}_t + \pi_t - \Delta m_t$. (A-3-47) ($\hat{l}_{H,t}$)

Similarly, we have:

$\hat{l}_{t-1}^* = \hat{l}_t^* + \pi_t^* - \Delta m_t^*$. (A-3-48) ($\hat{l}_{F,t}$)

3.9 Log-linearization of the Consolidated Government Budget Constraint

Eqs.(A-1-54) and (A-1-55) can be rewritten as:

$$\mathrm{B}_t = \frac{P_{H,t}}{P_t} G_t + \mathrm{B}_{t-1}(1+i_{t-1})\Pi_t^{-1} - TR_t - \frac{\Delta M_t}{P_t},$$

$$\mathrm{B}_t^* = \frac{P_{F,t}^*}{P_t^*} G_t^* + \mathrm{B}_{t-1}^*(1+i_{t-1}^*)(\Pi_t^*)^{-1} - TR_t^* - \frac{\Delta M_t^*}{P_t^*}.$$

Total derivative of the previous expressions yields:

$$d\mathrm{B}_t = dG_t + (1+\rho)d\mathrm{B}_{t-1} + \mathrm{B}d(1+i_{t-1}) - \mathrm{B}(1+\rho)d\Pi_t - dTR_t - d(\Delta M_t/P_t),$$

$$d\mathrm{B}_t^* = dG_t^* + (1+\rho)d\mathrm{B}_{t-1}^* + \mathrm{B}^* d(1+i_{t-1}^*) - \mathrm{B}^*(1+\rho)d\Pi_t^* - dTR_t^* - d(\Delta M_t^*/P_t^*).$$

where we use the fact that $G = 0$.

Dividing both sides of the previous expressions by Y yields:

$$\begin{aligned}\frac{d\mathrm{B}_t}{Y} &= \frac{dG_t}{Y} + (1+\rho)\frac{d\mathrm{B}_{t-1}}{Y} + bd(1+i_{t-1}) - b(1+\rho)d\Pi_t - \frac{dTR_t}{Y} \\ &- \frac{d(\Delta M_t/P_t)}{Y}\end{aligned}, \quad \text{(A-3-48)}$$

$$\begin{aligned}\frac{d\mathrm{B}_t^*}{Y} &= \frac{dG_t^*}{Y} + (1+\rho)\frac{d\mathrm{B}_{t-1}^*}{Y} + bd(1+i_{t-1}^*) - b(1+\rho)d\Pi_t^* - \frac{dTR_t^*}{Y} \\ &- \frac{d(\Delta M_t^*/P_t^*)}{Y}\end{aligned}. \quad \text{(A-3-49)}$$

where we use the definition $b \equiv \frac{\mathrm{B}}{Y}$.

Seignorage can be rewritten as:

$$\begin{aligned}\frac{\Delta M_t}{P_t} &= \frac{\Delta M_t}{M_{t-1}}\frac{P_{t-1}}{P_t}L_{t-1} \\ &= \frac{\Delta M_t}{M_{t-1}}\Pi_t^{-1}L_{t-1}\end{aligned},$$

$$\begin{aligned}\frac{\Delta M_t^*}{P_t^*} &= \frac{\Delta M_t^*}{M_{t-1}^*}\frac{P_{t-1}^*}{P_t^*}L_{t-1}^* \\ &= \frac{\Delta M_t^*}{M_{t-1}^*}(\Pi_t^*)^{-1}L_{t-1}^*\end{aligned}.$$

Total derivative of the previous expressions yields:

$$d(\Delta M_t/P_t) = \frac{L}{M}d\Delta M_t,$$

$$d\left(\Delta M_t^*/P_t^*\right)=\frac{L}{M}d\Delta M_t^*.$$

where we use the fact that $\Delta M=0$. Dividing both sides of the previous expressions yields by Y yields:

$$\frac{d\left(\Delta M_t/P_t\right)}{Y}=\chi\frac{d\Delta M_t}{M}, \text{ (A-3-50)}$$

$$\frac{d\left(\Delta M_t^*/P_t^*\right)}{Y}=\chi\frac{d\Delta M_t^*}{M}, \text{ (A-3-51)}$$

with $\chi\equiv\frac{L}{Y}$.

Plugging Eqs. (A-3-50) and (A-3-50) into Eqs.(A-3-48) and (A-3-49) yields:

$$\hat{b}_t=\hat{g}_t+(1+\rho)\hat{b}_{t-1}+(1+\rho)b\hat{i}_{t-1}-b(1+\rho)\pi_t-\widehat{tr}_t-\chi\Delta m_t, \text{ (A-3-52)}$$

$$\hat{b}_t^*=\hat{g}_t^*+(1+\rho)\hat{b}_{t-1}^*+(1+\rho)b\hat{i}_{t-1}^*-b(1+\rho)\pi_t^*-\widehat{tr}_t^*-\chi\Delta m_t^*, \text{ (A-3-53)}$$

with $\hat{b}_t\equiv\frac{d\mathrm{B}_t}{Y}$, $\hat{g}_t\equiv\frac{dG_t}{Y}$ and $\widehat{tr}_t\equiv\frac{TR_t-TR}{Y}$.

A simple tax rule in the non-FTPL is given by:

$$\widehat{tr}_t=\psi_b\hat{b}_{t-1}+\hat{\varsigma}_t, \text{ (A-3-54)}$$

$$\widehat{tr}_t^*=\psi_b\hat{b}_{t-1}^*+\hat{\varsigma}_t^*. \text{ (A-3-55)}$$

which is identical with Eq.(38) in the text.

3.10 Trade balance

Total derivative of Eq.(A-1-60) and is given by:

$$d\left(NX_t/P_{H,t}\right)=dY_t-C\left(\frac{dP_t}{P}-\frac{dP_{H,t}}{P_H}\right)-dC_t-dG_t.$$

By dividing both sides of the previous expression by Y yields:

$$\frac{d\left(NX_t/P_{H,t}\right)}{Y}=\frac{dY_t}{Y}-\left(\frac{dP_t}{P}-\frac{dP_{H,t}}{P_H}\right)-\frac{dC_t}{C}-\frac{dG_t}{Y},$$

which can be rewritten as:

$$\log\left(\frac{NX_t/P_{H,t}}{Y}\right)=\log\left(\frac{Y_t}{Y}\right)-\left[\log\left(\frac{P_t}{P}\right)-\log\left(\frac{dP_{H,t}}{P_H}\right)\right]-\log\left(\frac{C_t}{C}\right)-\log\left(\frac{G_t}{Y}\right).$$

Let define $\widehat{nx}_t \equiv \log\left[\frac{(NX_t/P_{H,t})}{Y}\right]$ and $\widehat{nx}_t^* \equiv \log\left[\frac{(NX_t^*/P_{H,t})}{Y^*}\right]$ which are the ratio of trade balance to the GDP. Then the previous expression can be rewritten as:

$$\widehat{nx}_t = \hat{y}_t - \left(\hat{p}_t - \hat{p}_{H,t}\right) - \hat{c}_t - \hat{g}_t .$$

Plugging Eqs.(A-3-10) and (A-3-11) yields:

$$\widehat{nx}_t = \hat{y}_t - (1-\nu)s_t - \hat{c}_t - \hat{g}_t . \text{ (A-3-58) } (\widehat{nx}_{H,t})$$

Eq.(A-1-61) can be rewritten as:

$$\begin{aligned}\frac{NX_t}{P_{F,t}^*} &= Y_t^* - g^*(S_t)C_t^* - G_t^* \\ &= Y_t^* - \frac{P_t^*}{P_{F,t}^*}C_t^* - G_t^*\end{aligned}.$$

Total derivative of the previous expression yields:

$$\begin{aligned}\frac{NX_t}{P_{F,t}^*} &= Y_t^* - \frac{P_t^*}{P_{F,t}^*}C_t^* - G_t^* \\ d\left(NX_t/P_{F,t}^*\right) &= dY_t^* - C^*\frac{dP_t^*}{P_F^*} + C^*\frac{dP_{F,t}^*}{P_F^*} - dC_t^* - dG_t^*\end{aligned}.$$

Dividing both sides of the previous expression yields:

$$\frac{d\left(NX_t/P_{F,t}^*\right)}{Y} = \frac{dY_t^*}{Y^*} - \frac{dP_t^*}{P_F^*} + \frac{dP_{F,t}^*}{P_F^*} - \frac{dC_t^*}{C^*} - \frac{dG_t^*}{Y^*},$$

which can be rewritten as:

$$\begin{aligned}\log\left(\frac{NX_t/P_{F,t}^*}{Y}\right) &= \log\left(\frac{Y_t^*}{Y^*}\right) - \log\left(\frac{P_t^*}{P_F^*}\right) + \log\left(\frac{P_{F,t}^*}{P_F^*}\right) - \log\left(\frac{C_t^*}{C^*}\right) - \log\left(\frac{G_t^*}{Y^*}\right) \\ &= \log\left(\frac{Y_t^*}{Y^*}\right) - \log P_t^* + \log P_{F,t}^* - \log\left(\frac{C_t^*}{C^*}\right) - \log\left(\frac{G_t^*}{Y^*}\right)\end{aligned}.$$

Let define $\widehat{nx}_t^* \equiv \log\left(\frac{NX_t / P_{F,t}^*}{Y}\right)$. Then:

$$\begin{aligned}\widehat{nx}_t^* &= \hat{y}_t^* - p_t^* + p_{F,t}^* - \hat{c}_t^* - \hat{g}_t^* \\ &= \hat{y}_t^* - \left[\nu p_{H,t}^* + (1-\nu) p_{F,t}^* - p_{F,t}^*\right] - \hat{c}_t^* - \hat{g}_t^* \\ &= \hat{y}_t^* - \left[-\nu\left(p_{F,t}^* - p_{H,t}^*\right)\right] - \hat{c}_t^* - \hat{g}_t^* \\ &= \hat{y}_t^* + \nu s_t - \hat{c}_t^* - \hat{g}_t^*\end{aligned} \quad \text{(A-3-59)}$$

Eq.(A-1-61) can be rewritten as:

$$\begin{aligned}\frac{NX_t}{P_{H,t}} &= -\frac{E_t P_{F,t}^*}{P_{H,t}} \frac{NX_t^*}{P_{F,t}^*} \\ &= -\frac{P_{F,t}}{P_{H,t}} \frac{NX_t^*}{P_{F,t}^*}\end{aligned}$$

Taking logarithm of the previous expression yields:

$$\log\left(\frac{NX_t}{P_{H,t}}\right) = -\log P_{F,t} + \log P_{H,t} + \log\left(\frac{NX_t^*}{P_{F,t}^*}\right).$$

Subtracting logY on both sides yields:

$$\begin{aligned}\log\left(\frac{NX_t / P_{H,t}}{Y}\right) &= -\left(\log P_{F,t} - \log P_{H,t}\right) + \log\left(\frac{NX_t^* / P_{F,t}^*}{Y^*}\right) \\ &= -\log S_t + \log\left(\frac{NX_t^* / P_{F,t}^*}{Y^*}\right)\end{aligned},$$

which can be rewritten as:

$$\widehat{nx}_t = -s_t + \widehat{nx}_t^*. \quad (\widehat{nx}_{F,t})$$

Plugging Eqs.(A-3-12) and (A-3-13) into Eqs.(A-3-58) and (A-3-59) yields:

$$\widehat{nx}_t = (\zeta - 1)(1-\nu) s_t,$$

$$\widehat{nx}_t^* = -(\zeta - 1)\nu s_t.$$

3.11 Iterated Government Budget Constraint

The government budget constraint is given by:

$P_{H,t}G_t + B_{t-1}(1+i_{t-1}) = P_t TR_t + B_t + \Delta M_t$. (A-3-60)

The level of seignorage, expressed as a fraction of steady state output can be approximated as:

$$\frac{\Delta M_t}{P_t}\frac{1}{Y} = \frac{\Delta M_t}{P_t}\frac{M_{t-1}}{M_{t-1}}\frac{P_{t-1}}{P_{t-1}}\frac{1}{Y}$$
$$= \frac{\Delta M_t}{M_{t-1}}\frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t}\frac{1}{Y} \text{. (A-3-62)}$$
$$= \frac{\Delta M_t}{M_{t-1}}\frac{P_{t-1}}{P_t}L_{t-1}\frac{1}{Y}$$

Quantity theory of money implies as follows:

$MV = PY$,

which can be rewritten as:

$V^{-1} = \frac{L}{Y}$.

Plugging the previous expression into Eq.(A-3-62) yields:

$\frac{\Delta M_t}{P_t}\frac{1}{Y} = \chi \Delta m_t$, (A-3-63)

with $\chi \equiv V^{-1}$ being the inverse of income velocity of money. Note that Eq.(A-3-63) ignore changes in the inflation and the deviation of the real money balance from its steady state.

If we do not ignore them, we have:

$$\frac{\Delta M_t}{P_t}\frac{1}{Y} = \frac{\Delta M_t}{M_{t-1}}\frac{P_{t-1}}{P_t}L_{t-1}\frac{1}{Y}$$
$$= \ln\left(\frac{M_t}{M_{t-1}}\right)\Pi_{t-1}^{-1}\frac{L_{t-1}}{L}\frac{L}{Y}$$
$$= \chi \ln\left(\frac{M_t}{M_{t-1}}\right)\Pi_{t-1}^{-1}\frac{L_{t-1}}{L}$$

Eq.(A-3-61) can be rewritten as:

$B_t + M_t = B_{t-1}(1+i_{t-1}) + M_{t-1} - P_t\left(TR_t - \frac{P_{H,t}}{P_t}G_t\right)$.

By applying $\frac{P_{H,t}}{P_t} = \frac{P_{H,t}}{P_{H,t}^{1-\nu} P_{F,t}^{\nu}} = S_t^{-\nu}$, the previous expression can be rewritten as:

$$B_t + M_t = B_{t-1}(1+i_{t-1}) + M_{t-1} - P_t\left(TR_t - S_t^{-\nu} G_t\right).$$

Let define $SP_t \equiv TR_t - S_t^{-\nu} G_t$. Then the previous expression can be rewritten as:

$$B_t + M_t = B_{t-1}(1+i_{t-1}) + M_{t-1} - P_t SP_t.$$

Previous equality can be rewritten as:

$B_t + M_t = (1+i_{t-1})[B_{t-1} + M_{t-1}] - i_{t-1} M_{t-1} - P_t SP_t$. (A-3-63)

Multiplying $(1+i_t)$ both sides of Eq.(A-3-63) yields:

$(1+i_t)(B_t + M_t) = (1+i_t)(1+i_{t-1})(B_{t-1} + M_{t-1}) - (1+i_t)(i_{t-1} M_{t-1} + P_t SP_t)$. (A-3-64)

Leading Eq.(A-3-64) one period or more ahead yields:

$(1+i_{t+1})(B_{t+1} + M_{t+1}) = (1+i_{t+1})(1+i_t)(B_t + M_t) - (1+i_{t+1})(i_t M_t + P_{t+1} SP_{t+1})$, (A-3-65)

$(1+i_{t+2})(B_{t+2} + M_{t+2}) = (1+i_{t+2})(1+i_{t+1})(B_{t+1} + M_{t+1}) - (1+i_{t+2})(i_{t+1} M_{t+1} + P_{t+2} SP_{t+2})$

, (A-3-66)

$(1+i_{t+3})(B_{t+3} + M_{t+3}) = (1+i_{t+3})(1+i_{t+2})(B_{t+2} + M_{t+2}) - (1+i_{t+3})(i_{t+2} M_{t+2} + P_{t+3} SP_{t+3})$.

(A-3-67)

Plugging Eq.(A-3-66) into Eq.(A-3-67) yields:

$$\begin{aligned}(1+i_{t+3})(B_{t+3} + M_{t+3}) = (1+i_{t+3})&\begin{bmatrix}(1+i_{t+2})(1+i_{t+1})(B_{t+1} + M_{t+1}) \\ -(1+i_{t+2})(i_{t+1} M_{t+1} + P_{t+2} SP_{t+2})\end{bmatrix}. \\ -(1+i_{t+3})&(i_{t+2} M_{t+2} + P_{t+3} SP_{t+3})\end{aligned}$$

Plugging Eq.(A-3-65) into the previous expression yields:

$$\begin{aligned}(1+i_{t+3})(B_{t+3} + M_{t+3}) = (1+i_{t+3})&\begin{Bmatrix}(1+i_{t+2})\begin{bmatrix}(1+i_{t+1})(1+i_t)(B_t + M_t) \\ -(1+i_{t+1})(i_t M_t + P_{t+1} SP_{t+1})\end{bmatrix} \\ -(1+i_{t+2})(i_{t+1} M_{t+1} + P_{t+2} SP_{t+2})\end{Bmatrix}. \\ -(1+i_{t+3})&(i_{t+2} M_{t+2} + P_{t+3} SP_{t+3})\end{aligned}$$

Plugging Eq.(A-3-64) into the previous expression yields:

$$
\begin{aligned}
(1+i_{t+3})(B_{t+3}+M_{t+3}) &= (1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(1+i_t)(1+i_{t-1})(B_{t-1}+M_{t-1}) \\
&\quad -(1+i_{t+3})\left\{\begin{matrix}(i_{t+2}M_{t+2}+P_{t+3}SP_{t+3}) \\ +(1+i_{t+2})\left[\begin{matrix}(i_{t+1}M_{t+1}+P_{t+2}SP_{t+2}) \\ +(1+i_{t+1})\left[\begin{matrix}(i_tM_t+P_{t+1}SP_{t+1}) \\ +(1+i_t)(i_{t-1}M_{t-1}+P_tSP_t)\end{matrix}\right]\end{matrix}\right]\end{matrix}\right\} \\
&= (1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(1+i_t)(1+i_{t-1})(B_{t-1}+M_{t-1}) \\
&\quad -(1+i_{t+3})(i_{t+2}M_{t+2}+P_{t+3}SP_{t+3}) \\
&\quad -(1+i_{t+3})(1+i_{t+2})(i_{t+1}M_{t+1}+P_{t+2}SP_{t+2}) \\
&\quad -(1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(i_tM_t+P_{t+1}SP_{t+1}) \\
&\quad -(1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(1+i_t)(i_{t-1}M_{t-1}+P_tSP_t)
\end{aligned}
.
$$

Iterating *j* times yields:

$$
\begin{aligned}
(1+i_{t+j})(B_{t+j}+M_{t+j}) &= (1+i_{t+j})(1+i_{t+j-1})\cdots(1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(1+i_t)(1+i_{t-1})(B_{t-1}+M_{t-1}) \\
&\quad -(1+i_{t+j})(i_{t+j-1}M_{t+j-1}+P_{t+j}SP_{t+j}) \\
&\quad -(1+i_{t+j})(1+i_{t+j-1})(i_{t+j-2}M_{t+j-2}+P_{t+j}SP_{t+j-1}) \\
&\quad -\cdots \\
&\quad -(1+i_{t+j})(1+i_{t+j-1})\cdots(1+i_{t+3})(i_{t+2}M_{t+2}+P_{t+3}SP_{t+3}) \\
&\quad -(1+i_{t+j})(1+i_{t+j-1})\cdots(1+i_{t+3})(1+i_{t+2})(i_{t+1}M_{t+1}+P_{t+2}SP_{t+2}) \\
&\quad -(1+i_{t+j})(1+i_{t+j-1})\cdots(1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(i_tM_t+P_{t+1}SP_{t+1}) \\
&\quad -(1+i_{t+j})(1+i_{t+j-1})\cdots(1+i_{t+3})(1+i_{t+2})(1+i_{t+1})(1+i_t)(i_{t-1}M_{t-1}+P_tSP_t)
\end{aligned}
$$

, which can be rewritten as:

$$
\begin{aligned}
(1+i_{t+j})(B_{t+j}+M_{t+j}) &= \prod_{h=0}^{j}(1+i_{t+h})(1+i_{t-1})(B_{t-1}+M_{t-1}) \\
&\quad -\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\right](i_{t+h-1}M_{t+h-1}+P_{t+h}SP_{t+h})
\end{aligned}
.
$$

Dividing both sides of the previous expression by P_{t+j+1} yields:

$$
\begin{aligned}
(1+i_{t+j})\frac{B_{t+j}+M_{t+j}}{P_{t+j+1}} &= \frac{1}{P_{t+j+1}}\prod_{h=0}^{j}(1+i_{t+h})(1+i_{t-1})(B_{t-1}+M_{t-1}) \\
&\quad -\frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\right](i_{t+h-1}M_{t+h-1}+P_{t+h}SP_{t+h})
\end{aligned}
. \quad \text{(A-3-68)}
$$

The first term in the RHS can be rewritten as:

$$
\begin{aligned}
&\frac{1}{P_{t+j+1}}\prod_{h=0}^{j}\left(1+i_{t+h}\right)\left(1+i_{t-1}\right)\left(B_{t-1}+M_{t-1}\right)\\
&=\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}\cdots\frac{P_{t+1}}{P_{t+2}}\frac{P_{t}}{P_{t+1}}\prod_{h=0}^{j}\left(1+i_{t+h}\right)\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}\\
&=\left[\prod_{h=0}^{j}\left(1+i_{t+h}\right)\frac{P_{t+h}}{P_{t+h+1}}\right]\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}\\
&=\left[\prod_{h=0}^{j}\left(1+i_{t+h}\right)\Pi_{t+h+1}^{-1}\right]\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}
\end{aligned}
\quad . \text{ (A-3-69)}
$$

The second term in the RHS can be rewritten as:

$$\frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left[\prod_{k=h}^{j}\left(1+i_{t+k}\right)\right]\left(i_{t+h-1}M_{t+h-1}+P_{t+h}SP_{t+h}\right)$$

$$=\frac{1}{P_{t+j+1}}\left[\begin{array}{l}\prod_{k=0}^{j}\left(1+i_{t+k}\right)\left(i_{t-1}M_{t-1}+P_tSP_t\right)+\prod_{k=1}^{j}\left(1+i_{t+k}\right)\left(i_tM_t+P_{t+1}SP_{t+1}\right)+\cdots\\+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)\left(i_{t+j-2}M_{t+j-2}+P_{t+j-1}SP_{t+j-1}\right)+\left(1+i_{t+j}\right)\left(i_{t+j-1}M_{t+j-1}+P_{t+j}SP_{t+j}\right)\end{array}\right]$$

$$=\frac{1}{P_{t+j+1}}\left[\begin{array}{l}\left(1+i_{t+j}\right)P_{t+j}SP_{t+j}+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)P_{t+j-1}SP_{t+j-1}+\cdots+\prod_{k=1}^{j}\left(1+i_{t+k}\right)P_{t+1}SP_{t+1}\\+\prod_{k=0}^{j}\left(1+i_{t+k}\right)P_tSP_t+\cdots\\\left(1+i_{t+j}\right)i_{t+j-1}M_{t+j-1}+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)i_{t+j-2}M_{t+j-2}+\cdots+\prod_{k=1}^{j}\left(1+i_{t+k}\right)i_tM_t\\+\prod_{k=0}^{j}\left(1+i_{t+k}\right)i_{t-1}M_{t-1}\end{array}\right]$$

$$=\left(1+i_{t+j}\right)\frac{P_{t+j}}{P_{t+j+1}}SP_{t+j}+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}SP_{t+j-1}+\cdots$$

$$+\prod_{k=1}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\cdots\frac{P_{t+2}}{P_{t+3}}\frac{P_{t+1}}{P_{t+2}}SP_{t+1}+\prod_{k=0}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\cdots\frac{P_{t+2}}{P_{t+3}}\frac{P_{t+1}}{P_{t+2}}\frac{P_t}{P_{t+1}}SP_t+\cdots$$

$$+\left(1+i_{t+j}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\left(i_{t+j-1}\right)\frac{M_{t+j-1}}{P_{t+j-1}}+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}\left(i_{t+j-2}\right)\frac{M_{t+j-2}}{P_{t+j-2}}+\cdots$$

$$+\prod_{k=1}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}\cdots\frac{P_{t+2}}{P_{t+3}}\frac{P_{t+1}}{P_{t+2}}\frac{P_t}{P_{t+1}}i_t\frac{M_t}{P_t}$$

$$+\prod_{k=0}^{j}\left(1+i_{t+k}\right)\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}\cdots\frac{P_{t+2}}{P_{t+3}}\frac{P_{t+1}}{P_{t+2}}\frac{P_t}{P_{t+1}}\frac{P_{t-1}}{P_t}i_{t-1}\frac{M_{t-1}}{P_{t-1}}$$

$$=\prod_{k=j}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}SP_{t+k}+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}SP_{t+k}+\cdots+\prod_{k=1}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}SP_{t+1}$$

$$+\prod_{k=0}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}SP_{t+k}+\cdots$$

$$+\prod_{k=j}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}\left(\Pi_{t+j}^{-1}i_{t+j-1}L_{t+j-1}\right)+\prod_{k=j-1}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}\left(\Pi_{t+j-1}^{-1}i_{t+j-2}L_{t+j-2}\right)+\cdots$$

$$+\prod_{k=1}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}\left(\Pi_{t+1}^{-1}i_tL_t\right)+\prod_{k=0}^{j}\left(1+i_{t+k}\right)\Pi_{t+k+1}^{-1}\left(\Pi_t^{-1}i_{t-1}L_{t-1}\right)$$

,

which can be rewritten as:

$$\frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\right](i_{t+h-1}M_{t+h-1}+P_{t+h}SP_{t+h})$$

$$=\prod_{k=j}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}SP_{t+k}+\prod_{k=j-1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}SP_{t+k}+\cdots+\prod_{k=1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}SP_{t+1}$$
$$+\prod_{k=0}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}SP_{t+k}+\cdots$$
$$+\prod_{k=j}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(\Pi_{t+j}^{-1}i_{t+j-1}L_{t+j-1}\right)+\prod_{k=j-1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(\Pi_{t+j-1}^{-1}i_{t+j-2}L_{t+j-2}\right)+\cdots$$
$$+\prod_{k=1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(\Pi_{t+1}^{-1}i_tL_t\right)+\prod_{k=0}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(\Pi_{t}^{-1}i_{t-1}L_{t-1}\right)$$
$$=\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]SP_{t+h}+\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]\left(\Pi_{t+h}^{-1}i_{t+h-1}L_{t+h-1}\right)$$

. (A-3-70)

Plugging Eqs.(A-3-69) and (A-3-70) into Eq.(A-3-68) yields:

$$(1+i_{t+j})\frac{B_{t+j}+M_{t+j}}{P_{t+j+1}}=\left[\prod_{h=0}^{j}(1+i_{t+h})\Pi_{t+h+1}^{-1}\right](1+i_{t-1})\frac{B_{t-1}+M_{t-1}}{P_t}$$
$$-\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]SP_{t+h}+\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]\left(\Pi_{t+h}^{-1}i_{t+h-1}L_{t+h-1}\right)$$

. (A-3-71)

Eq. (A-1-30) can be rewritten as:

$$\Pi_{t+1}^{-1}=\beta^{-1}\frac{1}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}$$. (A-3-72)

Plugging Eq.(A-3-72) into the first term in the RHS in Eq.(A-3-71) yields:

$$\left[\prod_{h=0}^{j}\left(1+i_{t+h}\right)\Pi_{t+h+1}^{-1}\right]\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$

$$=\Pi_{t+j+1}^{-1}\Pi_{t+j}^{-1}\Pi_{t+j-1}^{-1}\cdots\Pi_{t+2}^{-1}\Pi_{t+1}^{-1}\prod_{h=0}^{j}\left(1+i_{t+h}\right)\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$

$$=\left(\beta^{-1}\frac{1}{1+i_{t+j}}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}\right)\left(\beta^{-1}\frac{1}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}\right)\left(\beta^{-1}\frac{1}{1+i_{t+j-2}}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j-1}Z_{t+j-1}}\right)$$

$$\cdots\left(\beta^{-1}\frac{1}{1+i_{t+1}}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+2}Z_{t+2}}\right)\left(\beta^{-1}\frac{1}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}\right)\left(1+i_{t+j}\right)\left(1+i_{t+j-1}\right)\left(1+i_{t+j-2}\right)$$

$$\cdots\left(1+i_{t+1}\right)\left(1+i_t\right)\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$

$$=\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$

. (A-3-73)

Plugging Eq.(A-3-72) into the second term in the RHS in Eq.(A-3-71) yields:

$$\Pi_{t+1}^{-1}=\beta^{-1}\frac{1}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}$$

$$\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]SP_{t+h}+\sum_{h=0}^{j}\left[\prod_{k=h}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\right]\left(\Pi_{t+h}^{-1}i_{t+h-1}L_{t+h-1}\right)$$

$$=\prod_{k=j}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(SP_{t+j}+\Pi_{t+j}^{-1}i_{t+j-1}L_{t+j-1}\right)+\prod_{k=j-1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(SP_{t+j-1}+\Pi_{t+j-1}^{-1}i_{t+j-2}L_{t+j-2}\right)+$$

$$\cdots+\prod_{k=1}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(SP_{t+1}+\Pi_{t+1}^{-1}i_tL_t\right)+\prod_{k=0}^{j}(1+i_{t+k})\Pi_{t+k+1}^{-1}\left(SP_{t+0}+\Pi_t^{-1}i_{t-1}L_{t-1}\right)$$

$$=\left(1+i_{t+j}\right)\beta^{-1}\frac{1}{1+i_{t+j}}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}\left(SP_{t+j}+\beta^{-1}\frac{i_{t+j-1}}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}L_{t+j-1}\right)$$

$$+\left(1+i_{t+j}\right)\left(1+i_{t+j-1}\right)\beta^{-1}\frac{1}{1+i_{t+j}}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}$$

$$\times\beta^{-1}\frac{1}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}\left(SP_{t+j-1}+\beta^{-1}\frac{i_{t+j-2}}{1+i_{t+j-2}}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j-1}Z_{t+j-1}}L_{t+j-2}\right)+$$

$$\cdots+\left(1+i_{t+j}\right)\left(1+i_{t+j-1}\right)\cdots\left(1+i_{t+2}\right)\left(1+i_{t+1}\right)\beta^{-1}\frac{1}{1+i_{t+j}}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}\beta^{-1}\frac{1}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}$$

$$\cdots\beta^{-1}\frac{1}{1+i_{t+2}}\frac{U_{c,t+2}Z_{t+2}}{U_{c,t+3}Z_{t+3}}\beta^{-1}\frac{1}{1+i_{t+1}}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+2}Z_{t+2}}\left(SP_{t+1}+\beta^{-1}\frac{i_t}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}L_t\right)$$

$$+\left(1+i_{t+j}\right)\left(1+i_{t+j-1}\right)\cdots\left(1+i_{t+1}\right)\left(1+i_t\right)\beta^{-1}\frac{1}{1+i_{t+j}}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}\beta^{-1}\frac{1}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}$$

$$\cdots\beta^{-1}\frac{1}{1+i_{t+1}}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+2}Z_{t+2}}\beta^{-1}\frac{1}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}\left(SP_t+\beta^{-1}\frac{i_{t-1}}{1+i_{t-1}}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t}Z_t}L_{t-1}\right)$$

$$=\beta^{-1}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}\left(SP_{t+j}+\beta^{-1}\frac{i_{t+j-1}}{1+i_{t+j-1}}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j}Z_{t+j}}L_{t+j-1}\right)$$

$$+\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}\left(SP_{t+j-1}+\beta^{-1}\frac{i_{t+j-2}}{1+i_{t+j-2}}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j-1}Z_{t+j-1}}L_{t+j-2}\right)$$

$$+\cdots+\beta^{-j}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+j+1}Z_{t+j+1}}\left(SP_{t+1}+\beta^{-1}\frac{i_t}{1+i_t}\frac{U_{c,t}Z_t}{U_{c,t+1}Z_{t+1}}L_t\right)$$

$$+\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(SP_t+\beta^{-1}\frac{i_{t-1}}{1+i_{t-1}}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t}Z_t}L_{t-1}\right)$$

$$=\beta^{-1}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j}+\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-1}}{1+i_{t+j-1}}L_{t+j-1}+\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j-1}$$

$$+\beta^{-3}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-2}}{1+i_{t+j-2}}L_{t+j-2}+\cdots+\beta^{-j}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+1}+\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_t}{1+i_t}$$

$$+\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}SP_t+\beta^{-(j+2)}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}$$

. (A-3-74)

Plugging Eqs.(A-3-74) and (A-3-73) into Eq.(A-3-71) yields:

$$\left(1+i_{t+j}\right)\frac{B_{t+j}+M_{t+j}}{P_{t+j+1}}=\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$
$$-\left(\begin{array}{l}\beta^{-1}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j}+\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-1}}{1+i_{t+j-1}}L_{t+j-1}\\+\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j-1}+\beta^{-3}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-2}}{1+i_{t+j-2}}L_{t+j-2}\\+\cdots+\beta^{-j}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+1}+\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_t}{1+i_t}\\+\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}SP_t+\beta^{-(j+2)}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\end{array}\right)$$
$$=\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}$$
$$-\frac{1}{U_{c,t+j+1}Z_{t+j+1}}\left(\sum_{h=0}^{j}\beta^{h-j-1}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{j}\beta^{h-j-2}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\right)$$

Multiplying β^{t+j+1} on both sides of the previous expression yields:

$$
\begin{aligned}
&\beta^{t+j+1}\left(1+i_{t+j}\right)\frac{B_{t+j}+M_{t+j}}{P_{t+j+1}}=\beta^{t+j+1}\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}\\
&-\left(\begin{aligned}
&\beta^{t+j+1}\beta^{-1}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j}+\beta^{t+j+1}\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-1}}{1+i_{t+j-1}}L_{t+j-1}\\
&+\beta^{t+j+1}\beta^{-2}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j-1}+\beta^{t+j+1}\beta^{-3}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-2}}{1+i_{t+j-2}}L_{t+j-2}\\
&+\cdots+\beta^{t+j+1}\beta^{-j}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+1}+\beta^{t+j+1}\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_t}{1+i_t}\\
&+\beta^{t+j+1}\beta^{-(j+1)}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}SP_t+\beta^{t+j+1}\beta^{-(j+2)}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}
\end{aligned}\right)\\
&=\beta^{t}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}\\
&-\left(\begin{aligned}
&\beta^{t+j}\frac{U_{c,t+j}Z_{t+j}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j}+\beta^{t+j-1}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-1}}{1+i_{t+j-1}}L_{t+j-1}\\
&+\beta^{t+j-1}\frac{U_{c,t+j-1}Z_{t+j-1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+j-1}+\beta^{t+j-2}\frac{U_{c,t+j-2}Z_{t+j-2}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t+j-2}}{1+i_{t+j-2}}L_{t+j-2}\\
&+\cdots+\beta^{t+1}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t+j+1}Z_{t+j+1}}SP_{t+1}+\beta^{t}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_t}{1+i_t}\\
&+\beta^{t}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}SP_t+\beta^{t-1}\frac{U_{c,t-1}Z_{t-1}}{U_{c,t+j+1}Z_{t+j+1}}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}
\end{aligned}\right).\\
&=\beta^{t}\frac{U_{c,t}Z_t}{U_{c,t+j+1}Z_{t+j+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}\\
&-\frac{1}{U_{c,t+j+1}Z_{t+j+1}}\left(\sum_{h=0}^{j}\beta^{t+h}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{j}\beta^{t+h-1}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\right)
\end{aligned}
$$

Take the limit for $j\to\infty$ yields:

$$
\begin{aligned}
&0=\beta^{t}\frac{U_{c,t}Z_t}{U_{c,t+\infty+1}Z_{t+\infty+1}}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}\\
&-\frac{1}{U_{c,t+\infty+1}Z_{t+\infty+1}}\left(\sum_{h=0}^{\infty}\beta^{t+h}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{\infty}\beta^{t+h-1}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\right)
\end{aligned}
$$

Here, $\lim_{j\to\infty}\beta^{t+j+1}\left(1+i_{t+j}\right)\frac{B_{t+j}+M_{t+j}}{P_{t+j+1}}=0$ is the TVC.

Multiplying $U_{c,t+\infty+1}Z_{t+j+1}$ on both sides of the previous expression yields:

$$0=\beta^{t}U_{c,t}Z_{t}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}$$
$$-\left(\sum_{h=0}^{\infty}\beta^{t+h}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{\infty}\beta^{t+h-1}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\right).$$

Multiplying $\beta^{-(t+1)}$ on both sides of the previous expression yields:

$$0=\beta^{-(t+1)}\beta^{t}U_{c,t}Z_{t}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}$$
$$-\begin{pmatrix}\beta^{-(t+1)}\beta^{t+\infty}U_{c,t+\infty}Z_{t+\infty}SP_{t+\infty}+\beta^{-(t+1)}\beta^{t+\infty-1}U_{c,t+\infty-1}Z_{t+j-1}\frac{i_{t+\infty-1}}{1+i_{t+\infty-1}}L_{t+\infty-1}\\ +\beta^{-(t+1)}\beta^{t+\infty-1}U_{c,t+\infty-1}Z_{t+\infty-1}SP_{t+\infty-1}+\beta^{-(t+1)}\beta^{t+\infty-2}U_{c,t+\infty-2}Z_{t+\infty-2}\frac{i_{t+\infty-2}}{1+i_{t+\infty-2}}L_{t+\infty-2}\\ +\cdots+\beta^{-(t+1)}\beta^{t+1}U_{c,t+1}Z_{t+1}SP_{t+1}+\beta^{-(t+1)}\beta^{t}U_{c,t}Z_{t}\frac{i_{t}}{1+i_{t}}\\ +\beta^{-(t+1)}\beta^{t}U_{c,t}Z_{t}SP_{t}+\beta^{-(t+1)}\beta^{t-1}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\end{pmatrix}$$
$$=\beta^{-1}U_{c,t}Z_{t}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}$$
$$-\begin{pmatrix}\beta^{\infty-1}U_{c,t+\infty}Z_{t+\infty}SP_{t+\infty}+\beta^{\infty-2}U_{c,t+\infty-1}Z_{t+j-1}\frac{i_{t+\infty-1}}{1+i_{t+\infty-1}}L_{t+\infty-1}\\ +\beta^{\infty-2}U_{c,t+\infty-1}Z_{t+\infty-1}SP_{t+\infty-1}+\beta^{\infty-3}U_{c,t+\infty-2}Z_{t+\infty-2}\frac{i_{t+\infty-2}}{1+i_{t+\infty-2}}L_{t+\infty-2}\\ +\cdots+U_{c,t+1}Z_{t+1}SP_{t+1}+\beta^{-1}U_{c,t}Z_{t}\frac{i_{t}}{1+i_{t}}\\ +\beta^{-1}U_{c,t}Z_{t}SP_{t}+\beta^{-2}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\end{pmatrix},$$
$$=\beta^{-1}U_{c,t}Z_{t}\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_{t}}$$
$$-\left(\sum_{h=0}^{\infty}\beta^{h-1}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{\infty}\beta^{h-2}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\right)$$

which can be rewritten as:

$$\beta^{-1}U_{c,t}Z_t\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}=\begin{pmatrix}\beta^{\infty-1}U_{c,t+\infty}Z_{t+\infty}SP_{t+\infty}\\+\beta^{\infty-2}U_{c,t+\infty-1}Z_{t+j-1}\frac{i_{t+\infty-1}}{1+i_{t+\infty-1}}L_{t+\infty-1}\\+\beta^{\infty-2}U_{c,t+\infty-1}Z_{t+\infty-1}SP_{t+\infty-1}\\+\beta^{\infty-3}U_{c,t+\infty-2}Z_{t+\infty-2}\frac{i_{t+\infty-2}}{1+i_{t+\infty-2}}L_{t+\infty-2}\\+\cdots+U_{c,t+1}Z_{t+1}SP_{t+1}+\beta^{-1}U_{c,t}Z_t\frac{i_t}{1+i_t}\\+\beta^{-1}U_{c,t}Z_tSP_t+\beta^{-2}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\end{pmatrix}$$

$$=\sum_{h=0}^{\infty}\beta^{h-1}U_{c,t+h}Z_{t+h}SP_{t+h}+\sum_{h=0}^{\infty}\beta^{h-2}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\qquad\text{, (A-3-75)}$$

which can be rewritten as:

$$\beta^{-1}U_{c,t}Z_t\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}=\begin{pmatrix}\sum_{h=0}^{\infty}\beta^{h-1}U_{c,t+h}Z_{t+h}SP_{t+h}\\+\sum_{h=0}^{\infty}\beta^{h-2}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\end{pmatrix}$$

$$=\beta^{-1}U_{c,t}Z_tSP_t+U_{c,t+1}Z_{t+1}SP_{t+1}+\beta U_{c,t+2}Z_{t+2}SP_{t+2}+\cdots$$
$$+\beta^{-2}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}+\beta^{-1}U_{c,t}Z_t\frac{i_t}{1+i_t}L_t+U_{c,t+1}Z_{t+1}\frac{i_{t+1}}{1+i_{t+1}}L_{t+1}+\cdots\qquad\text{. (A-3-76)}$$

Leading the previous expression one period ahead yields:

$$\beta^{-1}U_{c,t+1}Z_{t+1}\left(1+i_t\right)\frac{B_t+M_t}{P_{t+1}}=\beta^{-1}U_{c,t+1}Z_{t+1}SP_{t+1}+U_{c,t+2}Z_{t+2}SP_{t+2}$$
$$+\beta U_{c,t+3}Z_{t+3}SP_{t+3}+\cdots+\beta^{-2}U_{c,t}Z_t\frac{i_t}{1+i_t}L_t+\beta^{-1}U_{c,t+1}Z_{t+1}\frac{i_{t+1}}{1+i_{t+1}}L_{t+1}\,.$$
$$+U_{c,t+2}Z_{t+2}\frac{i_{t+2}}{1+i_{t+2}}L_{t+2}+\cdots$$

Multiplying β on both sides of the previous expression yields:

$$U_{c,t+1}Z_{t+1}\left(1+i_t\right)\frac{B_t+M_t}{P_{t+1}}=U_{c,t+1}Z_{t+1}SP_{t+1}+\beta U_{c,t+2}Z_{t+2}SP_{t+2}$$

$$+\beta^2 U_{c,t+3}Z_{t+3}SP_{t+3}+\cdots+\beta^{-1}U_{c,t}Z_t\frac{i_t}{1+i_t}L_t+U_{c,t+1}Z_{t+1}\frac{i_{t+1}}{1+i_{t+1}}L_{t+1}$$

$$+\beta U_{c,t+2}Z_{t+2}\frac{i_{t+2}}{1+i_{t+2}}L_{t+2}+\cdots$$

Plugging the previous expression into Eq.(A-3-76) yields:

$$\begin{aligned}\beta^{-1}U_{c,t}Z_t\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}&=\beta^{-1}U_{c,t}Z_tSP_t+\beta^{-2}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\\&\quad+U_{c,t+1}Z_{t+1}\left(1+i_t\right)\frac{B_t+M_t}{P_{t+1}}\end{aligned}.$$

Multiplying β on both sides of the previous expression yields:

$$\begin{aligned}U_{c,t}Z_t\left(1+i_{t-1}\right)\frac{B_{t-1}+M_{t-1}}{P_t}&=U_{c,t}Z_tSP_t+\beta^{-1}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\\&\quad+\beta U_{c,t+1}Z_{t+1}\left(1+i_t\right)\frac{B_t+M_t}{P_{t+1}}\end{aligned}. \quad \text{(A-3-77)}$$

Eq.(A-3-77) can be rewritten as:

$$\begin{aligned}U_{c,t}Z_t\left(1+i_{t-1}\right)\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_t^{-1}&=U_{c,t}Z_tSP_t+\beta^{-1}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}\\&\quad+\beta U_{c,t+1}Z_{t+1}\left(1+i_t\right)\left(\mathrm{B}_t+L_t\right)\Pi_{t+1}^{-1}\end{aligned}, \quad \text{(A-3-78)}$$

with $\mathrm{B}_t\equiv\frac{B_t}{P_t}$.

Eq.(A-3-75) can be rewritten as:

$$U_{c,t}Z_t\left(1+i_{t-1}\right)\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_t^{-1}=\left(\begin{aligned}&\sum_{h=0}^{\infty}\beta^hU_{c,t+h}Z_{t+h}SP_{t+h}\\&+\sum_{h=0}^{\infty}\beta^{h-1}U_{c,t+h-1}Z_{t+h-1}\frac{i_{t+h-1}}{1+i_{t+h-1}}L_{t+h-1}\end{aligned}\right), \quad \text{(A-3-79)}$$

Similarly, we have:

$$U_{c,t}^{*}Z_{t}^{*}\left(1+i_{t-1}^{*}\right)\left(\mathrm{B}_{t-1}^{*}+L_{t-1}^{*}\right)\left(\Pi_{t}^{*}\right)^{-1}=U_{c,t}^{*}Z_{t}^{*}SP_{t}^{*}+\beta^{-1}U_{c,t-1}^{*}Z_{t-1}^{*}\frac{i_{t-1}^{*}}{1+i_{t-1}^{*}}L_{t-1}^{*} +\beta U_{c,t+1}^{*}Z_{t+1}^{*}\left(1+i_{t}^{*}\right)\left(\mathrm{B}_{t}^{*}+L_{t}^{*}\right)\left(\Pi_{t+1}^{*}\right)^{-1}, \quad \text{(A-3-80)}$$

3.12 Log-linearizing Iterated Government Budget Constraint

Eq.(A-3-78) can be rewritten as:

$$\beta^{-1}U_{c,t-1}Z_{t-1}\frac{i_{t-1}}{1+i_{t-1}}L_{t-1}=U_{c,t}Z_{t}\left(1+i_{t-1}\right)\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_{t}^{-1} -\beta U_{c,t+1}Z_{t+1}\left(1+i_{t}\right)\left(\mathrm{B}_{t}+L_{t}\right)\Pi_{t+1}^{-1}-U_{c,t}Z_{t}SP_{t}.$$

Multiplying $\beta\frac{1}{U_{c,t-1}Z_{t-1}}\left(1+i_{t-1}\right)\frac{1}{L_{t-1}}$ on both sides of the previous expression yields:

$$\begin{aligned} i_{t-1}=&\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t-1}\right)^{2}\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_{t}^{-1}L_{t-1}^{-1}\\ &-\beta^{2}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t}\right)\left(1+i_{t-1}\right)\left(\mathrm{B}_{t}+L_{t}\right)\Pi_{t+1}^{-1}L_{t-1}^{-1}-\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}SP_{t}\left(1+i_{t-1}\right)L_{t-1}^{-1}. \end{aligned}$$

Summing 1 on both sides yields:

$$\begin{aligned} 1+i_{t-1}=&\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t-1}\right)^{2}\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_{t}^{-1}L_{t-1}^{-1}\\ &-\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}SP_{t}\left(1+i_{t-1}\right)L_{t-1}^{-1}\\ &-\beta^{2}\frac{U_{c,t+1}Z_{t+1}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t}\right)\left(1+i_{t-1}\right)\left(\mathrm{B}_{t}+L_{t}\right)\Pi_{t+1}^{-1}L_{t-1}^{-1}+1\\ =&\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t-1}\right)^{2}\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_{t}^{-1}L_{t-1}^{-1}\\ &-\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}SP_{t}\left(1+i_{t-1}\right)L_{t-1}^{-1}\\ &-\beta\frac{U_{c,t+1}Z_{t+1}}{U_{c,t}Z_{t}}\left(1+i_{t}\right)\beta\frac{U_{c,t}Z_{t}}{U_{c,t-1}Z_{t-1}}\left(1+i_{t-1}\right)\left(\mathrm{B}_{t}+L_{t}\right)\Pi_{t+1}^{-1}L_{t-1}^{-1}+1 \end{aligned}.$$

Note that $SP_{t}\equiv TR_{t}-S_{t}^{-\nu}G_{t}$. Plugging Eq.(A-3-72)

into the previous expression yields:

$$
\begin{aligned}
1+i_{t-1} &= \frac{1}{1+i_{t-1}}\Pi_t\left(1+i_{t-1}\right)^2\left(\mathrm{B}_{t-1}+L_{t-1}\right)\Pi_t^{-1}L_{t-1}^{-1} \\
&-\frac{1}{1+i_{t-1}}\Pi_t SP_t\left(1+i_{t-1}\right)L_{t-1}^{-1} \\
&-\frac{1}{1+i_t}\Pi_{t+1}\left(1+i_t\right)\frac{1}{1+i_{t-1}}\Pi_t\left(1+i_{t-1}\right)\left(\mathrm{B}_t+L_t\right)\Pi_{t+1}^{-1}L_{t-1}^{-1}+1 \\
&=\frac{1+i_{t-1}}{1+i_{t-1}}\left(1+i_{t-1}\right)\Pi_t\Pi_t^{-1}\left(\mathrm{B}_{t-1}+L_{t-1}\right)L_{t-1}^{-1} \\
&-\frac{1+i_{t-1}}{1+i_{t-1}}\Pi_t SP_t L_{t-1}^{-1} \\
&-\frac{1+i_t}{1+i_t}\frac{1+i_{t-1}}{1+i_{t-1}}\Pi_{t+1}\Pi_{t+1}^{-1}\Pi_t\left(\mathrm{B}_t+L_t\right)L_{t-1}^{-1}+1 \\
&=\left(1+i_{t-1}\right)\left(\frac{\mathrm{B}_{t-1}}{L_{t-1}}+1\right)-\Pi_t SP_t L_{t-1}^{-1}-\Pi_t\left(\mathrm{B}_t+L_t\right)L_{t-1}^{-1}+1
\end{aligned}
$$

Total derivative of the previous expression is given by:

$$
\begin{aligned}
d\left(1+i_{t-1}\right) &= \left(\frac{\mathrm{B}}{L}+1\right)d\left(1+i_{t-1}\right)+\frac{1+\rho}{L}d\mathrm{B}_{t-1} \\
&\quad +\left[-(1+\rho)\mathrm{B}L^{-2}+SPL^{-2}+(\mathrm{B}+L)L^{-2}\right]dL_{t-1} \\
&\quad -\frac{SP}{L}\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{SP}{L}+\left(\frac{\mathrm{B}}{L}+1\right)\right]d\Pi_t \\
&= \left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)d\left(1+i_{t-1}\right)+(1+\rho)\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y} \\
&\quad +\left[-(1+\rho)\frac{\mathrm{B}}{Y}\frac{Y}{L}+(1+\rho)\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)\right]\frac{dL_{t-1}}{L} \\
&\quad -\frac{Y}{L}\frac{\mathrm{B}}{Y}\left(\frac{1-\beta}{\beta}\right)\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t \\
&= \left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)d\left(1+i_{t-1}\right)+(1+\rho)\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y} \\
&\quad +(1+\rho)\left[-\frac{\mathrm{B}}{Y}\frac{Y}{L}+\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\beta\frac{Y}{L}\frac{\mathrm{B}}{Y}+\beta\right]\frac{dL_{t-1}}{L} \\
&\quad -\frac{Y}{L}\frac{\mathrm{B}}{Y}\left(\frac{1-\beta}{\beta}\right)\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t \\
&= \left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)d\left(1+i_{t-1}\right)+(1+\rho)\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y} \\
&\quad +(1+\rho)\left[-\frac{\mathrm{B}}{Y}\frac{Y}{L}(1-\beta)+\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\beta\right]\frac{dL_{t-1}}{L} \\
&\quad -\frac{Y}{L}\frac{\mathrm{B}}{Y}\left(\frac{1-\beta}{\beta}\right)\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t \\
&= \left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)d\left(1+i_{t-1}\right)+(1+\rho)\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y} \\
&\quad +(1+\rho)\left[-\frac{\mathrm{B}}{Y}\frac{Y}{L}(1-\beta)\left(1-\frac{1}{\beta}\right)+\beta\right]\frac{dL_{t-1}}{L} \\
&\quad -\frac{Y}{L}\frac{\mathrm{B}}{Y}\left(\frac{1-\beta}{\beta}\right)\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t \\
&= \left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)d\left(1+i_{t-1}\right)+(1+\rho)\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y} \\
&\quad +(1+\rho)\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}(1-\beta)\left(\frac{1-\beta}{\beta}\right)+\beta\right]\frac{dL_{t-1}}{L} \\
&\quad -\frac{Y}{L}\frac{\mathrm{B}}{Y}\left(\frac{1-\beta}{\beta}\right)\frac{dSP_t}{SP}-\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\frac{dL_t}{L}-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}\left(\frac{1-\beta}{\beta}\right)+\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t
\end{aligned}
$$

.

Dividing both sides of the previous expression by $1+\rho$ yields:

$$\frac{d\left(1+i_{t-1}\right)}{1+\rho}=\left(\frac{Y}{L}\frac{\mathrm{B}}{Y}+1\right)\frac{d\left(1+i_{t-1}\right)}{1+\rho}+\frac{Y}{L}\frac{d\mathrm{B}_{t-1}}{Y}$$
$$+\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}(1-\beta)\left(\frac{1-\beta}{\beta}\right)+\beta\right]\frac{dL_{t-1}}{L}-\frac{Y}{L}\frac{\mathrm{B}}{Y}(1-\beta)\frac{dSP_t}{SP}-\beta\frac{Y}{L}\frac{d\mathrm{B}_t}{Y}-\beta\frac{dL_t}{L},$$
$$-\left[\frac{\mathrm{B}}{Y}\frac{Y}{L}(1-\beta)+\beta\left(\frac{\mathrm{B}}{Y}\frac{Y}{L}+1\right)\right]d\Pi_t$$

which can be rewritten as:

$$\hat{i}_{t-1}=\left(\frac{b}{\chi}+1\right)\hat{i}_{t-1}+\frac{1}{\chi}\hat{b}_{t-1}+\left[\frac{b(1-\beta)^2}{\chi\beta}+\beta\right]\hat{l}_{t-1}$$
$$-\frac{b(1-\beta)}{\chi}\widehat{sp}_t-\frac{\beta}{\chi}\hat{b}_t-\beta\hat{l}_t-\left[\frac{b(1-\beta)}{\chi}+\beta\frac{b}{\chi}+\beta\right]\pi_t$$
$$=\frac{b+\chi}{\chi}\hat{i}_{t-1}+\frac{1}{\chi}\hat{b}_{t-1}+\frac{b(1-\beta)^2+\chi\beta^2}{\chi\beta}\hat{l}_{t-1} \quad ,$$
$$-\frac{b(1-\beta)}{\chi}\widehat{sp}_t-\frac{\beta}{\chi}\hat{b}_t-\beta\hat{l}_t-\left(\frac{b}{\chi}+\beta\right)\pi_t$$
$$=\frac{b+\chi}{\chi}\hat{i}_{t-1}+\frac{1}{\chi}\hat{b}_{t-1}+\frac{b(1-\beta)^2+\chi\beta^2}{\chi\beta}\hat{l}_{t-1}-\frac{b(1-\beta)}{\chi\beta}\widehat{sp}_t-\frac{\beta}{\chi}\hat{b}_t-\beta\hat{l}_t-\frac{b+\chi\beta}{\chi}\pi_t$$

with $b\equiv\frac{\mathrm{B}}{Y}$, $\chi\equiv\frac{L}{Y}$, $\widehat{sp}_t\equiv\frac{dSP_t}{SP}$, $\hat{\xi}_t\equiv\frac{dU_{c,t}}{U_c}$, $\hat{l}_t\equiv\frac{dL_t}{L}$, $\hat{i}_t\equiv\frac{d\left(1+i_t\right)}{1+\rho}$, $\pi_t\equiv d\Pi_t$,

$\hat{b}_t\equiv\frac{d\mathrm{B}_t}{Y}$ and $\hat{\rho}_t\equiv-\log\left(\frac{Z_{t+1}}{Z_t}\right)$.

The previous expression can be rewritten as:

$$\hat{i}_{t-1}+\frac{b(1-\beta)}{\chi\beta}\widehat{sp}_t=\frac{b+\chi}{\chi}\hat{i}_{t-1}+\frac{1}{\chi}\hat{b}_{t-1}+\frac{b(1-\beta)^2+\chi\beta^2}{\chi\beta}\hat{l}_{t-1}-\frac{\beta}{\chi}\hat{b}_t-\beta\hat{l}_t-\frac{b+\chi\beta}{\chi}\pi_t$$

The LHS is revenue which consists of interest payment deprived from households and the fiscal surplus. The first to the third terms in the RHS is expenditure which consists of burden to redeem government debt with interest payment and real money. The fourth to fifth terms mitigate that burden. Newly issued government debt and real money and inflation tax.

Further:

$$\frac{b(1-\beta)}{\chi\beta}\widehat{sp}_t = \frac{b}{\chi}\hat{i}_{t-1} + \frac{1}{\chi}\hat{b}_{t-1} + \frac{b(1-\beta)^2+\chi\beta^2}{\chi\beta}\hat{l}_{t-1} - \frac{\beta}{\chi}\hat{b}_t - \beta\hat{l}_t - \frac{b+\chi\beta}{\chi}\pi_t .$$

Here, seignorage in the LHS disappear. That is canceled by interest payment.

Plugging $\widehat{sp}_t = \frac{\beta}{b(1-\beta)}\widehat{tr}_t - \frac{\beta}{b(1-\beta)}\hat{g}_t$ into the previous expression yields:

$$\frac{1}{\chi}\widehat{tr}_t - \frac{1}{\chi}\hat{g}_t = \frac{b}{\chi}\hat{i}_{t-1} + \frac{1}{\chi}\hat{b}_{t-1} + \frac{b(1-\beta)^2+\chi\beta^2}{\chi\beta}\hat{l}_{t-1} - \frac{\beta}{\chi}\hat{b}_t - \beta\hat{l}_t - \frac{b+\chi\beta}{\chi}\pi_t$$

which can be rewritten as:

$$\widehat{tr}_t = b\hat{i}_{t-1} + \hat{b}_{t-1} + \frac{b(1-\beta)^2+\chi\beta^2}{\beta}\hat{l}_{t-1} - \beta\hat{b}_t - \beta\chi\hat{l}_t - (b+\chi\beta)\pi_t + \hat{g}_t . \quad \text{(A-3-81)}$$

which is a class of log-linearized Euler equation.
Similarly, we ha:

$$\widehat{tr}_t^* = b\hat{i}^*_{t-1} + \hat{b}^*_{t-1} + \frac{b(1-\beta)^2+\chi\beta^2}{\beta}\hat{l}^*_{t-1} - \beta\hat{b}^*_t - \beta\chi\hat{l}^*_t - (b+\chi\beta)\pi^*_t + \hat{g}^*_t . \quad \text{(A-3-82)}$$

Log-lineraizing the definition of $SP_t \equiv TR_t - S_t^{-\nu}G_t$ yields:

$$\begin{aligned} dSP_t &= dTR_t - (-\nu)GdS_t - dG_t \\ &= dTR_t - dG_t \end{aligned} .$$

Dividing both sides of the previous expression by SP yields:

$$\begin{aligned} \frac{dSP_t}{SP} &= \frac{1}{\mathrm{B}}\left(\frac{\beta}{1-\beta}\right)dTR_t - \frac{1}{\mathrm{B}}\left(\frac{\beta}{1-\beta}\right)dG_t \\ &= \frac{Y}{\mathrm{B}}\left(\frac{\beta}{1-\beta}\right)\frac{dTR_t}{Y} - \frac{Y}{\mathrm{B}}\left(\frac{\beta}{1-\beta}\right)\frac{dG_t}{Y} \\ &= \frac{\beta}{b(1-\beta)}\frac{dTR_t}{Y} - \frac{\beta}{b(1-\beta)}\frac{dG_t}{Y} \end{aligned} .$$

Let define $\widehat{tr}_t \equiv \frac{dTR_t}{Y}$ and $\hat{g}_t \equiv \frac{dG_t}{Y}$. The n the previous expression can be rewritten as:

$$\widehat{sp}_t = \frac{\beta}{b(1-\beta)}\widehat{tr}_t - \frac{\beta}{b(1-\beta)}\hat{g}_t .$$

3.13 Log-linearizing Iterated Government Budget Constraint

Eq.(A-1-38) can be rewritten as:

$$U_{c,t} Z_t = U^*_{c,t} Z^*_t .$$

where we use Eqs.(A-1-20) and (A-1-25). The previous expression can be rewritten as:

$$U_{c,t} = U^*_{c,t} \frac{Z^*_t}{Z_t} .$$

Following Okano and Eguchi (2023), we assume no persistence on Preference shock. Then:

$$U_{c,t} = U^*_{c,t} \frac{Z_{t+1}}{Z_t} \frac{Z^*_t}{Z^*_{t+1}},$$

where $Z_{t+1} = Z^*_{t+1} = 1$. Log-linearizing the previous expression yields:

$$\log U_{c,t} = \log U^*_{c,t} + \log\left(\frac{Z_{t+1}}{Z_t}\right) - \log\left(\frac{Z^*_{t+1}}{Z^*_t}\right).$$

Subtracting $\log U_c$ from both sides of the previous expression yields:

$$\log\left(\frac{U_{c,t}}{U_c}\right) = \log\left(\frac{U^*_{c,t}}{U^*_c}\right) + \log\left(\frac{Z_{t+1}}{Z_t}\right) - \log\left(\frac{Z^*_{t+1}}{Z^*_t}\right),$$

which can be rewritten as:

$$\hat{\xi}_t = \hat{\xi}^*_t - \hat{\rho}_t + \hat{\rho}^*_t . \text{ (A-3-83)}$$

3.13 Rewriting the Market Clearing and Euler Equation

Eq.(A-3-12) can be rewritten as:

$$\hat{y}^*_t = \hat{y}_t - s_t - \hat{g}_t + \hat{g}^*_t , \text{ (A-3-91)}$$

$$\hat{y}_t = \hat{y}^*_t + s_t + \hat{g}_t - \hat{g}^*_t . \text{ (A-3-92)}$$

Plugging Eq.(A-3-91) into Eq.(A-3-13) yields:

$$\nu\hat{y}_t + (1-\nu)(\hat{y}_t - s_t - \hat{g}_t + \hat{g}_t^*) = \nu\hat{c}_t + (1-\nu)\hat{c}_t^* + \nu\hat{g}_t + (1-\nu)\hat{g}_t^*,$$

which can be rewritten as:

$$\hat{y}_t = \nu\hat{c}_t + (1-\nu)s_t + (1-\nu)\hat{c}_t^* + \hat{g}_t. \text{ (A-3-93)}$$

Plugging Eq.(A-3-92) into Eq.(A-3-13) yields:

$$\hat{y}_t^* = \nu\hat{c}_t - \nu s_t + (1-\nu)\hat{c}_t^* + \hat{g}_t^*. \text{ (A-3-94)}$$

Subtracting Eq.(A-3-16) from Eq.(A-3-15) yields:

$$\hat{\xi}_t - \hat{\xi}_t^* = \hat{\xi}_{t+1} - \hat{\xi}_{t+1}^* + \hat{i}_t - \hat{i}_t^* - \pi_{t+1} + \pi_{t+1}^* - \hat{\rho}_t + \hat{\rho}_t^*.$$

Plugging Eq.(A-3-83) into the previous expression yields:

$$\hat{i}_t = \hat{i}_t^* + e_{t+1} - e_t + \hat{\rho}_{t+1} - \hat{\rho}_{t+1}^*. \text{ (A-3-95)}$$

Eqs.(A-3-93), (A-3-94) and (A-3-95) can replace Eqs.(A-3-12), (A-3-13) and (A-3-83)

3.14 Some Entities

The PPP

$$e_t = p_t - p_t^*. \text{ (A-3-84)}$$

The CPI Inflation

$$\pi_t = \pi_{H,t} + (1-\nu)\Delta s_t \text{ (A-3-85)}$$

$$\pi_t^* = \pi_{F,t}^* - \nu\Delta s_t \text{ (A-3-86)}$$

Changes in the CPI Inflation

$$\pi_t = p_t - p_{t-1}, \text{ (A-3-87)}$$

$$\pi_t^* = p_t^* - p_{t-1}^*. \text{ (A-3-88)}$$

Import Inflation

$$\pi_{F,t} = \pi_{F,t}^* + e_t - e_{t-1}, \text{ (A-3-89)}$$

$$\pi_{H,t}^* = \pi_{H,t} - e_t + e_{t-1}. \text{ (A-3-90)}$$

4 Policy Regimes

Plugging $\hat{b}_t = 0$ for all *t* into Eq.(A-3-56) yields:

$$\Delta m_t = \frac{1}{\chi}\hat{g}_t + \frac{(1+\rho)b}{\chi}\hat{i}_{t-1} - \frac{(1+\rho)b}{\chi}\pi_t - \frac{1}{\chi}\widehat{tr}_t \text{, (A-3-52)}$$

5

The domestic and the imported goods inflation is given by:

6.5.4 The Steady State in the Case of No Subsidiary

We focus on equilibria where the state variables follow paths that are close to a deterministic stationary equilibrium, in which $\Pi_{H,t} = \Pi_t = 1$. Further, we assume $Z_t = Z_t^* = 1$ and $G_t = 0$.

Eqs.(A-1-30) and (A-1-14) implies as follows:

$$\beta = \frac{1}{1+i} = \frac{1}{1+i^*},$$

which is identical with Eq.(A-2-1).

Eq.(A-1-32) implies that:

$$\frac{W}{P} = \frac{V_n}{U_c},$$

which is identical with Eq.(A-2-2).

Eq.(A-1-34) implies as follows:

$$\frac{U_l}{U_c} = \beta i,$$

which is identical with Eq.(A-2-4).

Eq.(6-4) implies:

$$1 = \frac{\frac{\varepsilon}{\varepsilon-1}(1-\tau_F)\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^*\right)^{-1}\right]^{-1} C_F}{\left[1+\theta\beta+(\theta\beta)^2+\cdots\right]\left[\left(U_c^*\right)^{-1}\right]^{-1} C_F \Psi^{-1}},$$

which can be rewritten as:

$$\Psi = \left[\mathrm{M}(1-\tau_F)\right]^{-1},$$

with $\mathrm{M} \equiv \frac{\varepsilon}{\varepsilon - 1}$ being the constant markup. As long as we assume $\tau_F = 0$,

$\Psi = \mathrm{M}^{-1}$. (6-15)

Eq.(A-1-46) implies:

$$MC = \frac{1}{1-\alpha}\frac{W}{P_H}N^{\alpha},$$

Which is identical with Eq.(2-5).

Eq.(A-1-5) can be rewritten as:

$$\frac{V_n}{U_c} = \frac{W}{P_H}\frac{P_H}{P},$$

which is identical with Eq.(2-6).

Plugging Eq.(2-5) into Eq.(2-6) yields:

$$\frac{V_n}{U_c} = \frac{1-\alpha}{N^{\alpha}\mathrm{M}}\frac{P_H}{P},$$

which is identical with Eq.(2-7).

Plugging Eq.(2-8) into Eq.(2-7) yields:

$$\frac{V_n}{U_c} = \frac{1-\alpha}{N^{\alpha}\mathrm{M}S^{\nu}},$$

which can be written as:

$$V_n = \frac{1-\alpha}{N^{\alpha}\mathrm{M}S^{\nu}}U_c,$$

which is identical with Eq.(A-2-16)

Eq.(6-1) implies:

$$U_c^{-1} = \vartheta\left(U_c^*\right)^{-1} S^{1-\nu}\Psi.$$

Plugging Eq.(6-15) into the previous expression yields:

$$\begin{aligned} U_c^{-1} &= \vartheta\left(U_c^*\right)^{-1} S^{1-\nu}\mathrm{M}^{-1} \\ &= \vartheta\left(U_c^*\right)^{-1} \omega(S) \end{aligned}. \quad (6\text{-}16)$$

Note that:

$$\begin{aligned} \omega(S) &\equiv Q \\ &= \frac{EP^*}{P} = \frac{P_F}{P_H^{1-\nu}P_F^{\nu}}\frac{EP_F^*}{P_F} = \left(\frac{P_F}{P_H}\right)^{1-\nu}\mathrm{M}^{-1} \\ &= S^{1-\nu}\mathrm{M}^{-1} \end{aligned}. \quad (6\text{-}17)$$

Eq.(6-16) can be rewritten as:

$$S^{\nu} = \vartheta\left(U_c^*\right)^{-1} S\mathrm{M}^{-1}U_c$$

Plugging the previous expression into Eq.(A-2-16) yields:

$$\begin{aligned} V_n &= \frac{1-\alpha}{N^{\alpha}}\frac{\mathrm{M}^{-1}}{\vartheta\left(U_c^*\right)^{-1} S\mathrm{M}^{-1}} \\ &= \frac{1-\alpha}{N^{\alpha}}\frac{1}{\vartheta\left(U_c^*\right)^{-1} S} \end{aligned}. \text{ (6-18)}$$

Let define $H\left(S,U_c^*\right) \equiv V_n N^{\alpha}$. Plugging this definition into Eq.(6-18) yields:

$$H\left(S,U_c^*\right) \equiv (1-\alpha)\frac{1}{S\vartheta\left(U_c^*\right)^{-1}}.$$

Notice that $H_S < 0$, $\lim_{S\to 0} H\left(S,U_c^*\right) = +\infty$ and $\lim_{S\to\infty} H\left(S,U_c^*\right) = 0$ ($g_S > 0$).

On the other hand, the market clearing Eq.(6-6) implies:

$Y = (1-\nu)S^{\nu}C + \nu\mathrm{M}^{-1}SY^*$, (6-19)

where we use $C^* = Y^*$.

Because of $C = F\left(U_c^{-1}\right)$ and Eq.(6-16), we have:

$$\begin{aligned} C &= F\left[\vartheta\left(U_c^*\right)^{-1}\omega(S)\right] \\ &= F\left[\vartheta\left(U_c^*\right)^{-1} S^{1-\nu}\mathrm{M}^{-1}\right] \end{aligned},$$

with *F* being the operator of function.

Plugging the previous expression into Eq.(6-19) yields:

$Y = (1-\nu)S^{\nu}F\left[\vartheta\left(U_c^*\right)^{-1} S^{1-\nu}\mathrm{M}^{-1}\right] + \nu SC^*$. (6-20)

Let define $J\left(S,C^*\right) \equiv (1-\nu)S^{\nu}F\left[\vartheta\left(U_c^*\right)^{-1} S^{1-\nu}\mathrm{M}^{-1}\right] + \nu SY^*$. Note that $J_S > 0$, $\lim_{S\to 0} J\left(S,C^*\right) = 0$ and $\lim_{S\to\infty} J\left(S,C^*\right) = +\infty$.

Hence, given a value for C^*, ϑ and Y^*, Eqs.(6-18) and (6-20), jointly determine the steady state value for *S* and $\omega(S)$, i.e., the steady state value of the TOT and the real exchange rate.

Dividing both sides of Eq.(6-19) by C^* yields:

$$\frac{Y}{C^*}=(1-\nu)S^{\nu}\frac{C}{C^*}+\nu\mathrm{M}^{-1}S.$$

For convenience, and without loss of generality, we can assume that initial conditions (i.e., initial distribution of wealth) are such that $\vartheta=1$ which implies that $Q=\frac{C}{C^*}$.

Plugging this condition into the previous expression yields:

$$\begin{aligned}\frac{Y}{C^*}&=(1-\nu)S^{\nu}Q+\nu S\\&=(1-\nu)S^{\nu}S^{1-\nu}\Psi+\nu S\\&=[(1-\nu)\Psi+\nu]S\end{aligned}$$

where we use a steady state condition $Q=S^{1-\nu}\Psi$ which stems from Eq.(A.1).
which can be rewritten as:

$Y=[(1-\nu)\Psi+\nu]SY^*$, (6-20)

by using $Y^*=C^*$ which is the steady state market clearing condition in the foreign country. Eq.(2-15) is no longer applicable.
Eqs.(2-17)—(2-20) is still applicable. Thus. Eq.(2-21), i.e., $S=1$ is applicable. However, even if plugging Eq.(2-21) into Eq.(6-20), we cannot obtain Eq.(2-15) because $\Psi=1$ is not applicable.
Plugging Eq.(2-21) into a steady state condition $Q=S^{1-\nu}\Psi$ yields:
$Q=\mathrm{M}^{-1}$, (6-21)
where we use Eq.(6-15). The PPP in the long run is no longer available.
Plugging Eq.(6-21) into the initial condition yields:
$C=C^*\mathrm{M}^{-1}$, (6-22)
That is, $C=C^*$ is no longer available.
Plugging Eq.(6-22) into Eq.(6-20) yields:

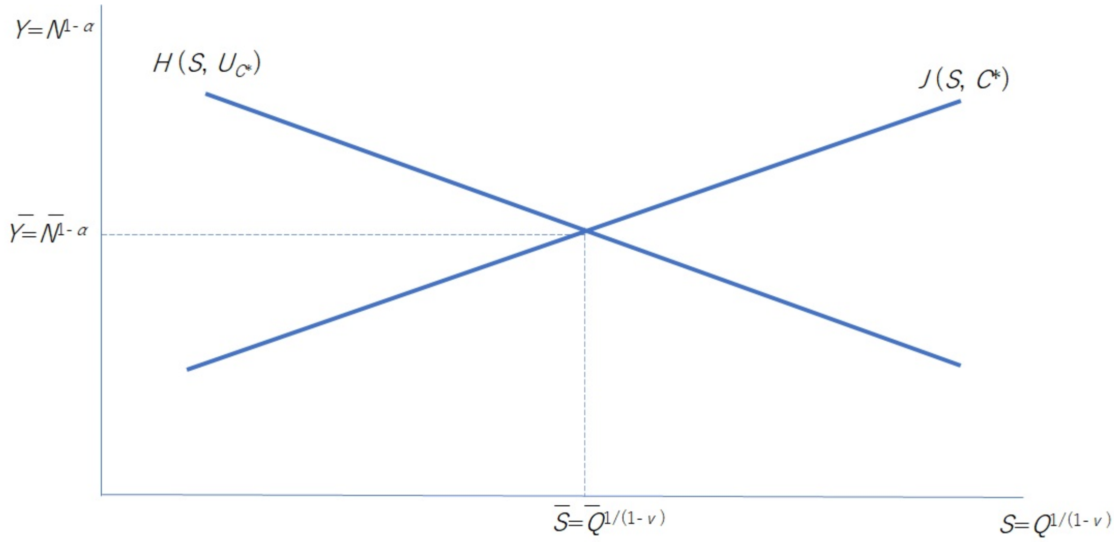
$$\begin{aligned}Y&=[(1-\nu)\mathrm{M}^{-1}+\nu]C\mathrm{M}\\&=(1-\nu)C+\nu C\mathrm{M}\\&=[(1-\nu)+\nu\mathrm{M}]C\end{aligned}.$$

Thus, $Y=C$ is no longer available.

Reference (Not shown in the text only)

Gali, Jordi (2015), ``Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Application (2nd Eds.),'' *Princeton University Press*, New York.

Figure TA--1



(Relative) PPP in a Small Open Economy in the Steady State